7. IMPACT OF SEASONAL LEVEL SHIFT (SLS) ON TIME SERIES FORECASTING

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Abstract

The effect of not treating Seasonal Level Shift (SLS) outliers on forecast accuracy, and prediction intervals is the focus of this study. We examine the impact of SLS on point and interval forecasts using simulation experiment for time series models including SAR (1) and SMA (1) for different parameter values, sample sizes and time of occurrences. We extend the strategy suggested by Asghar and Urooj (2017) to forecasting in the presence of SLS by looking at forecast accuracy and prediction interval. We demonstrate that SLS significantly increases the inaccuracy of the SARIMA models, increases the bias in the SARIMA estimates, and significantly affects the prediction intervals. However, after detection and adjustment of SLS, SARIMA estimates become less biased, and forecast accuracy measure and prediction interval significantly improve. The difference of location of SLS from forecast origin has similar effect on bias and forecast accuracy in SAR (1) model. While, in SMA (1) model, the SLS occurring at the beginning of the series has greater adverse effect than that occurring at the middle or end of the series. Three monthly time series data from Pakistan are used to explore the issue.

Keywords: Seasonal Level Shift (SLS), SARIMA, forecast accuracy, point forecasts, interval forecasts

JEL Classification: C15, C18, C63, C32, C87

1. Introduction

Mostly time series data exhibit the problem of large disturbance as well as structural changes, where outliers and structural changes cause misleading analysis such as inappropriate model selection, improper decomposition of the series, biased parameter estimation and importantly misleading forecasts (Chen & Liu; 1993). Time series forecasting is a key technique used at all levels for effective policy and strategic decision-making under uncertainty. The contamination of time series with outliers and breaks results in distortion of forecasts. In such cases, the forecast accuracy is reduced due to the biased estimates of

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the model parameters and will have long lasting effect on the outliers. As noticed by Cousineau & Chertier, (2010), some outliers are at times enough to distort the whole analysis and make it a necessity to be dealt with. The detection of outliers is important for all disciplines (such as economics, finance, physics, public health and machine learning along with others). According to Bollen & Jackman (1985), the outliers have a greater impact on statistical significance when sample size is small. Similarly, Chen (2001) observed that outliers in experimental data lead to inccurate interpretation of the results of fuzzy linear regression. Therefore, many researchers identify the presence of outliers and use strategies for detection of these large disturbances in time series data. The literature shows that the presence of outliers in historical data can occasionally has a significant impact on forecast accuracy. Box & Jenkins (1973), Tsay (1988), Chen & Liu (1993a, 1993b), Balke (1994), Kaiser & Maravall (1999) and many other scholars examined the types and structure of outliers and their effects on diagnostics, model specification and forecasting. The effect of one time shock (additive outlier) on forecasts was explored by Hillmer (1984) and Ledolter (1989). Stock & Watson (1996, 1999) noticed that ignorance of outliers misleads forecasts in addition to other reasons.

Outliers are observations that are unusually smaller or much larger than the rest of the data (Bollen & Jackman; 1985). In simple words, outliers in time series are those data that do not match the typical observations and depart from the regular trend and/or seasonal component pattern. The type and location of outliers determine the deviation of observation from the trend. Fox (1972) identified few different types of outliers and suggested their detection method. Though Chang (1982) and Tsay (1986) and many others examined two types of outliers, Tsay (1988) and Chen & Liu (1993) identified four different types of outliers namely level shift (LS), transient change (TC), innovative outlier (IO), and additive outlier (AO). Kaiser & Maravall (2001) proposed and explored seasonal outliers. Several researchers investigated the existence, impact and detection of outliers in various time series designs, including Nair *et al.* (2006), Mustafa (2009), Urooj & Asghar (2017), Asghar & Urooj (2017). Many studies examined and argued about the effect of nature, magnitude, and timing of various type of outliers. As Tsay (1988), Chen & Liu (1993), Balke & Fomby (1994) & Maravall (2007) noted biased parameter estimates due to outliers. According to Charles (2004, 2006) the presence of outliers led to non-normality, excess kurtosis and skewness.

Economic forecasting has long been a point of interest, although, it has a long history of systematic forecast failure, despite all known methodologies. According to Hendry & Ericsson (2001), the precense of shifts and discontinuities interact with other issues and exacerbate the forecast failure. Hence, it is essential to identify and analyse the outliers before forecasting. Many econometric methods for times series forecasting have been proposed; however, limited progress has been seen in identifying and monitoring the impact of outliers in time series forecasting, especially in the case of Seasonal Level Shift (SLS).

This motivates us to examine the consequences of outliers such as SLS and other types on forecasts. The studies by Urooj (2016), Urooj & Asghar (2017), Asghar & Urooj (2017) and Urooj & Asghar (2020) examined the existence, impact and detection of AO, IO, LS, TC and SLS in time series data for various SARIMA(p,d,q)$(P,D,Q)_s$[4] models also by collecting empirical evidence from time series data for Pakistan. However, these studies did not focus

[4] *Seasonal Autoregressive Moving Average (SARIMA (p, d, q)(P, D, Q)s) Model, where s, p, d, q, P, D, Q are seasonal frequency order, non-seasonal and seasonal roots, order of integration and seasonal integration of the model as defined by Box and Jenkins (1976). AR(p), MA(q), SAR(P) and SMA(Q) are the special form models defined under SARIMA (p, d, q)$(P, D, Q)_s$.*

on the forecasting performance in the presence of outliers. The current study focuses on the forecasting of time series models in the presence of SLS. We attempt to answer three questions: How does the unrecognized seasonal level shift (SLS) affect the width of the prediction intervals? What is the impact of SLS on forecast for different sample sizes? And how does the impact of outlier vary due to the distance of outlier from the forecast origin? To answer these, we explore the influence of SLS on forecasts in SARIMA models. We have examined the impact of SLS on point and interval forecast in terms of forecast accuracy through simulation for free of SLS, with SLS and adjusted for SLS series using the sampling distribution of coefficients of SAR(1) and SMA(1), also written as SARIMA(0,0,0)(1,0,0)$_s$ models, respectively, where 's' indicates the seasonal frequency. This study also examines the carry - over effects of seasonal level shift on forecasts by looking at the impact of time of occurrence of SLS along with the exploration of impact of outlier for different sample sizes. The empirical study is also conducted for Pakistan, using variables measured on monthly frequency.

Section 2 is about analytical framework, followed by description of the outliers, outlier detection method and simulation strategy. The simulation results of the impact of SLS on forecast accuracy and prediction interval for various time series models is explored in Section 3. Section 4 presents empirical analysis for Pakistan using monthly time series data with outliers. Section 5 concludes with a brief discussion on the implications of the results.

2. Analytical Framework

Studies including Fox (1972), Chang *et al.* (1988), Tsay (1988), Chen & Liu (1993a, 1993b), Vaage (2000), Kaiser & Maravall (2001), Hotta *et.al* (2004), Urooj & Asghar (2017) and Asghar and Urooj (2017) emphasized on the performance of univariate time series analysis in the presence of outliers. Bollen and Jackman (1985) argues the vital influence of outliers in case of sample size and less robust statistics. The studies by Urooj (2016), Urooj and Asghar (2017), Asghar and Urooj (2017) and Urooj and Asghar (2020) examined the existence, impact and detection of AO, IO, LS, TC and SLS in time series data for various SARIMA(p,d,q)(P,D,Q)$_s$[5] models.

In this study, seasonal ARIMA model is used to investigate the effects of the outliers especially SLS on the performance of time series forecasts. A seasonal ARIMA model for quarterly and monthly series is usually indicated as SARIMA (p, d, q) (P, D, Q)$_s$. Consider an outlier free time series y_t such that

$$\phi(B)\Phi(B^s)\nabla^d\nabla_s^D y_t = \theta(B)\Theta(B^s)a_t \tag{1}$$

where: B is backshift lag operator as $By_t = y_{t-1}$ while 's' shows the seasonal frequency i.e., s=4 or 12 for quarterly or monthly series respectively and 'd' is the order of integration. The other terms are defined as

$$\phi(B) = 1 - \phi_1 B^1 - \phi_2 B^2 \ldots \phi_p B^p,$$

$$\Phi(B^s) = 1 - \Phi_1 \mathrm{B}^s - \cdots \Phi_p B^{sp},$$

$$\Theta(B^s) = 1 - \Theta_1 \mathrm{B}^s - \cdots \Theta_q B^{sq}$$

[5] *Seasonal Autoregressive Moving Average (SARIMA (p, d, q)(P, D, Q)$_s$) Model, where s, p, d, q, P, D, Q are seasonal frequency order, non-seasonal and seasonal roots, order of integration and seasonal integration of the model as defined by Box and Jenkins (1976). AR(p), MA(q), SAR(P) and SMA(Q) are the special form models defined under SARIMA (p, d, q)(P, D, Q)$_s$.*

$\theta(B) = 1 - \theta_1 B^1 - \theta_2 B^2 \ldots \theta_q B^q$

With roots lying outside the unit circle, $\nabla^d \nabla_s^D y_t$ is the stationary series with $a_t \sim N(0, \sigma_a^2)$. We may define

$\pi(\mathrm{B}) = \frac{\nabla^d \nabla_s^D \phi(B)\Phi(B^s)}{\theta(B)\Theta(B^s)} = 1 - \pi_1 B - \pi_2 B^2 \ldots$

Such that equation (1) can be written as $\frac{\phi(\mathrm{B})\Phi(B^s)}{\theta(B)\Theta(B^s)} \nabla^d \nabla_s^D y_t = a_t$

or equivalently, $\pi(\mathrm{B})\, y_t = a_t$ (2)

Equation (2) forms an AR (∞) process. It can also be written as

$$y_t = \psi(B) a_t \quad (3)^6$$

$where: \psi(B) = \frac{\theta(B)\Theta(B^s)}{\phi(B)\Phi(B^s)\nabla^d \nabla_s^D} = \frac{(1-\theta_1 B - \theta_2 B^2 \ldots - \theta_q B^q)(1 - \Theta_1 \mathrm{B}^s - \cdots \Theta_q B^{sq})}{(1-B)^d (1-B_s^D)(1-\phi_1 B - \phi_2 B^2 \ldots - \phi_p B^p)(1 - \Phi_1 \mathrm{B}^s - \cdots \Phi_p B^{sp})}$.

Due to the existence of outliers the series is unobservable and the observed series is contaminated with outliers as (see Fox; 1972, Bell and Hillmer; 1983, Tsay; 1988)

$$z_t = y_t + A_t \quad (2)$$

where: A_t is a parametric function, may be deterministic or stochastic depending on the type of disturbance, representing the exogenous disturbances of z_t and is given as

$$A_t = \omega_i v_i(B) I_t^{(T)}$$

where: $I_t^{(T)}$ is an indicator variable such that $I_t^{(T)} = 1\ at\ t = T\ and\ zero\ elsewhere,\ \omega_i$ is the magnitude of i[th] outlier, $v_i(B)$ determines the dynamics of outliers for i=SLS. The series z_t is a linear combination of the stationary series y_t and the parametric function A_t. Hence, z_t forms stationary process with roots lying outside the unit circle (Box and Jenkins, 1970, section 3.2 and Chatfield, 2016, section 3.4)

Asghar and Urooj (2017) investigated five types of outliers, *i.e*., additive outlier (AO), innovative outlier (IO), level shift (LS), transient change (TC) and seasonal level shift (SLS) but they did not examine the impact of SLS on point and interval forecast A brief overview of these five types of outliers is given as

i) *Additive Outlier (AO)* is an outlier (external/exogenous change) which occurs at a particular time t_0, affects one observation only. Its effects are independent of the models, but has serious effects on parameter estimates and forecasts. ω_A is the magnitude of AO, $\nu_A(B)$ determines the dynamic of outliers $\nu_A(B) = 1$.

ii) *Innovative outlier (IO)* is an internal but aberrant innovation affecting the observed time series for some time span after the occurrence date. The effects of the innovation outlier are less serious in estimation and inferences. ω_{IO} is the magnitude of IO, $\nu_{IO}(B)$ determines the dynamic effect of outliers as $\nu_{IO}(B) = \frac{1}{\pi(B)}$.

iii) *Level shift (LS)* is a permanent change in the time series. The level shift effect is large in stationary process as compared to non-stationary process. ω_{LS} is the magnitude of LS, $\nu_{LS}(B)$ determines the dynamic of LS as $\nu_{LS}(B) = \frac{1}{(1-B)}$.

iv) *Transitory change (TC)* is the level shifts which dies out exponentially. ω_{TC} is the magnitude of TC, $\nu_{TC}(B)$ determines the dynamic impact of TC as $\nu_{TC}(B) = \frac{1}{(1-\delta B)}$ where δ is the decaying parameter determining the rate of gradual decline in impact of TC.

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[6] *var (y)=*$\sigma^2\ [\omega(B)]^2$

v) *Seasonal Level shift (SLS)* is the interference which upsets only certain quarters or months of the years. It is the special kind of level shift which arises in SARIMA(p,d,q)(P,D,Q)$_s$ at some point T such that for $1 \leq T \leq n$, in time and re-occur regularly every year at same season say S and its effect carries up to subsequent seasons. ω_{SLS} is the magnitude of SLS, $v_{SLS}(B) = \frac{1}{\nabla_s} - \frac{1}{s\nabla}$ determines the dynamic impact of SLS $v_{SLS}(B) = \frac{1}{\nabla_s} - \frac{1}{s\nabla}$ $where\ \nabla = 1 - B\ \ and\ \nabla_s = (1 - B^s)$.

For detection and adjustment of outliers, iterative procedure suggested by Kaiser & Maravall (2001) and revised by Asghar & Urooj (2017) is used which consists of three stages. In stage one, first step is to compute the maximum likelihood estimates (MLE) of the model parameters using an initial SARIMA(P, D, Q) model on actual observed series assuming that there are no outliers, then obtain the residuals. The observed series from eq (4) is re-written as

$$z_t = \psi(B)a_t + \omega_i v_i(B) I_t^{(T)}, \text{ or}$$
$$\pi(\mathrm{B})\, z_t = a_t + \omega_i v_i(B) I_t^{(T)}, \text{ or}$$
$$Z_t = \omega_i x_t + u_t \tag{5}$$

The estimates of outliers' magnitude and effect along with their variances are obtained using MLE and are given by $\widehat{\omega_i} = \frac{\sum_{t=1}^{n} z_t x_t}{\sum_{t=1}^{n} x_t^2}$ and variance as $Var(\widehat{\omega_i}) = \frac{\sigma_u^2}{\sum_{t=1}^{n} x_t^2}$ for i=AO, IO, LS, TC, SLS. In step two of stage one, compute test statistics for different outliers using the residuals. These test statistics are constructed using likelihood ratio for testing the existence of outlier at time point d as $\lambda_{i,d} = \frac{\widehat{\omega}_{i,d}}{\sqrt{Var(\widehat{\omega_i})}}$; $i = \mathrm{AO, IO, LS, TC, SLS}$. Computing maximum of the test statistics and comparing $\{\lambda_{IO,max}, \lambda_{AO,max}, \lambda_{TC,max}, \lambda_{LS,max}, \lambda_{SLS,max}\}$ with critical value c[7], if $\lambda_{i,max} < C$, at pre-decided cut off points $C = \{3, 3.5, 4\}$, then there is no significant outliers, and if $\lambda_{i,max} \geq c$ there is significant outlier. If no outliers are found then stop and conclude, otherwise, adjust the effect from the residuals as well as the observed series z_t is adjusted at time $t = \tau_1$ to obtain the corrected y_t via equation (5) using the estimated magnitude $\widehat{\omega_i}$ and the appropriate dynamic impact of outlier and repeat this procedure until no more outliers are found. In stage two run a multiple regression model for joint estimation of outlier's effect using the residuals as an output variable and the first stage's identified outliers as the input variables. Calculate the test statistics for probable outliers and if found significant, remove the outliers from the set of identified outliers and run again the estimation cycle of multiple regression until no outlier is found. Now obtain the adjusted series by removing the significant outlier effects then compute the MLE for model parameters using adjusted series and repeat the whole process for further iteration until the relative change of the residuals become negligible. In the final stage, run the intervention model with estimated parameters and significant outliers (for detail see Asghar & Urooj (2017) and Kaiser & Maravall (2001)). This intervention model is used for forecasting purpose.

Estimating the initial ARIMA(p, d, q) model can lead to misidentifying level shifts as innovational outliers or cannot detect them at all (Darné & Charles, 2011). To better

[7] *These three values are selected based on simulation results of Chang (1982) and provided satisfactory results, also see Chang and Tiao (1983) and Tsay (1986).*

determine whether the outliers can be considered as permanent or not, an outlier search will be conducted using the series in levels, *i.e*., from an ARIMA(p, 0, q) (Balke & Fomby 1991; Balke 1993).The proposed iterative procedure is less vulnerable to spurious and masking effects during outlier detection and allows to jointly estimate the model parameters and multiple outlier effects (Dagum & Bianconcini, 2010). The estimation of intervention model allows to verify if there are any insignificant lags to be removed from the model.

2.1 Mathematical Exploration of Impact of SLS on Forecast

We intend to explore the impact of SLS on forecast through various SARIMA models. This section investigates mathematically the impact of SLS on forecast through SARIMA(0,0,0)(1,0,0)$_4$ and SARIMA(0,0,0)(0,0,1)$_4$ model in the presence of SLS. The specification of model lags is via automatic selection process based on minimum AIC. The analysis comprises of model identification, parameter estimation, point forecasts and interval forecasts. We initiate by considering an outliers free time series y_t then the contaminated series is considered and the forecast is evaluated by measuring the mean square of the h-step ahead forecast error MSFE (h;k, ω) and the relative increase in this mean square error that is due to SLS at any time T(I MSFE (h;k, ω)).

2.2 Simulation Strategy

We study the impact of SLS on point and interval forecast of various time series specifications by observing the behaviour of SARIMA(0,0,0)(1,0,0)$_4$ and SARIMA(0,0,0)(0,0,1)$_4$ model, respectively, under three different scenarios, namely series free of SLS, series affected by the presence of a SLS and series adjusted for SLS. Simulation experiment is run for the choice of parameter values $\Phi_4 = \{0.2, 0.4, 0.6, 0.8)$ for SARIMA (0,0,0)(1,0,0)$_4$ or SAR(1), as we shall call it now onwards, and $\Theta_4 = \{0.2, 0.4, 0.6, 0.8)$ for SARIMA(0,0,0)(0,0,1)$_4$ or SMA(1) model, as we shall call it now onwards. We have examined the impact of SLS of size $\omega_{SLS} = 5\sigma$ on series of size $n = \{50, 100\}$ for 5000 iterations. We focus on the impact of SLS at different locations by examining the performance of the forecast in terms of forecast accuracy for point forecast using different measures of aggregate error namely Mean Error (ME), Root Mean Square Error (RMSE), Mean Absolute Error (MAE) and Mean Absolute Standard Error (MASE). For interval forecast we look at the length of the prediction intervals using data generating process of SARIMA, we generate three series, *i.e.,* with SLS, without SLS, and adjusted for SLS, then estimate the model, and get parameter estimates, and their standard errors.

Forecast accuracy/error is the difference between the actual values and the forecast value for a given time period $E_t = Y_t - F_t$, where E is the forecast error of given time, Y_t is the actual value and F_t is the forecast value. Forecast accuracy is calculated using ME, MASE, RMSE, MAE for point and interval one step ahead forecast. Different measure of aggregate error used are:

I. Mean Error (ME) calculated as $ME = \frac{\sum_{t=1}^{N} e_t}{N}$,

II. Root Mean Square Error (RMSE) as $\mathrm{RMSE} = \sqrt{\frac{\sum_{t=1}^{N} e_t^2}{N}}$,

III. Mean Absolute Error (MAE) as $MAE = \frac{\sum_{t=1}^{N} |e_t|}{N}$ and

IV. Mean Absolute Scale Error (MASE) as $MASE = \frac{\sum_{t=1}^{N} \left| \frac{e_t}{\frac{1}{N-m} \sum_{t=m+1}^{N} |Y_t - Y_{t-m}|} \right|}{N}$, Where m= seasonal period or 1 if non-seasonal.

2.3 Empirical Study

The studies by Urooj (2016), Urooj & Asghar (2017), Asghar & Urooj (2017) and Urooj & Asghar (2020) examined the existence, impact and detection of AO, IO, LS, TC and SLS in time series data for various SARIMA(p,d,q)(P,D,Q)$_s$[8] models also by collecting empirical evidence from time series data for Pakistan. Further, the influence of outliers is significant for small sample size and less robust statistics (Bollen & Jackman; 1985). However, these studies did not focus on the forecasting performance in the presence of outliers. The current study focuses on the forecasting of time series models in the presence of SLS.

Lack of empirical literature for impact of outliers, especially of SLS on forecast in the case of Pakistan motivates us to empirically explore the impact on forecast in the presence of SLS in case of monthly time series data for Pakistan. Three monthly series were considered, namely Tax collection (2004 M1 to 2016 M6) collected by Federal Board of Revenue, Money in circulation (2002 M1 to 2016 M12) and Broad money (2006 M10 to 2016 M12). The data is taken from Federal Board of Revenue (FBR) annual reports, and International Financial Statistics (IFS). The assessment regarding outliers and structural breaks is conducted using Kaiser & Maravall (2001)'s suggested iterative procedure for multiple outlier detection and joint estimation was modified by Asghar & Urooj (2017) as discussed in earlier section.

3. Impact of SLS on Forecasts

Seasonal shift is a special kind of level shift which occurs in SARIMA(p, d, q)(P,D,Q)$_s$ at some point T for $1 \leq T \leq n$ in time and reoccurs regularly every year at the same season, say S_j $j = 0,1,2, \ldots$ for seasonal frequency of S and its effects caries up to $(s-1)$ subsequent seasons. We attempt to derive the resulting increase in the Mean Square Error of the L-step ahead forecast. This increase is due to two effects: (i) the carry over effect of outlier on the forecast and (ii) the bias in the estimates of the coefficients of SARIMA models. As noted by Pena (2001), the forecast uncertainty is due to three sources of variability amongst which one is model uncertainty including the impact of outliers as well. Suppose the outlier occurs at time point $t = T$such that $n = ls = T + k$ is the length of time series. Suppose we (the forecaster) do not observe the series y_t directly as it is contaminated with outlier, namely SLS, and hence have ignored its adjustment in estimation of coefficients and calculation of forecast errors. We are now attempting to study the impact of SLS on forecast from the SARIMA models.

3.1. Effects of SLS on the Forecast from SARIMA Model with Known Coefficient

To better understand the impact of SLS on forecast we consider the model from (1) is outlier free series

$$\Phi(B^s)\phi(B)(1-B)(1-B^s)y_t = \theta(B)\Theta(B^s)a_t$$

While observed series carrying the impact of outlier is given by,

$z_t = y_t + A_t$, such that

[8] *Seasonal Autoregressive Moving Average (SARIMA (p, d, q)(P, D, Q)$_s$) Model, where s, p, d, q, P, D, Q are seasonal frequency order, non-seasonal and seasonal roots, order of integration and seasonal integration of the model as defined by Box and Jenkins (1976). AR(p), MA(q), SAR(P) and SMA(Q) are the special form models defined under SARIMA (p, d, q)(P, D, Q)$_s$.*

$$z_t = y_t + \omega_{SLS}\, S(B) I_t(T) \qquad (6)$$

where: $S(B) = \frac{S}{S-1}\left[\frac{1}{1-B^S} - \frac{1}{S(1-B)}\right] = \left(\frac{S}{S-1}\right)\left(\frac{1}{1-B^S}\right) - \frac{1}{(S-1)(1-B)}$

$$= (1 + B^s + B^{2s} + \cdots) - \frac{1}{S-1}(B + B^2 + B^3 + \cdots B^{S+r} \ldots)$$

$$S(B) = \sum_{j=0}^{ls-T} B^{jS} - \frac{1}{S-1}\sum_{j=0}^{ls-T}\sum_{i=0}^{S-1} B^{i+jS}$$

where $ls = n = T + k$ Hence equation (6) becomes:

$$z_t = y_t + \omega_{SLS}\left(\sum_{j=0}^{ls-T} B^{jS} - \frac{1}{S-1}\sum_{j=0}^{ls-T}\sum_{i=0}^{S-1} B^{i+jS}\right) I_t(T) \qquad (7)$$

for $I_t^T = \begin{matrix} 1 & if\ t = T \\ 0 & if\ t \neq T \end{matrix}$ is the variable representing the presence/absence of outliers, ω_{SLS} is the effect of outlier.

Let us assume that coefficients of SARIMA model are known and outlier at point T is ignored. The one-step ahead forecast when the outlier occurs at the last point observed, *i.e.*, T=n. It was established under Chen and Liu (1993b) that the type of outlier would not be determined and it would be like additive outlier, but it is ignored. So, the forecast would be

$$z_n(1) = \pi_1 z_n + \pi_2 z_{n-1} + \cdots$$

$$\hat{z}_T(1) = \pi_1 z_T + \pi_2 z_{T-1} + \cdots$$

where: $\hat{z}_T$ is the 1-step ahead forecast at time T, z_t is the observed data point with π_j as weights

$$\hat{z}_T(1) = \pi_1(y_T + \omega_{SLS}) + \pi_2 z_{T-1} + \cdots$$

Since $z_{T-1}, z_{T-2}, \ldots$are already outlier free, so,

$$\hat{z}_T(1) = \pi_1 y_T + \pi_1\omega_{SLS} + \pi_2 y_{T-1} + \cdots$$

$$\hat{z}_T(1) = \pi_1\hat{y}_T(1) + \pi_1\omega_{SLS} \qquad (8)$$

With $\sum_{j=1}^{k}\pi_j = 1$ and the 1-step ahead forecast error is $a_{T+1} = z_{T+1} - \hat{z}_T(1)$. Now for h-step ahead minimum mean square error forecast made at time t=n=T is

$$\hat{z}_T(h) = \pi_1{}^{(h)} z_T + \pi_2{}^{(h)} z_{T-1} + \cdots$$

where: $\pi_j^{(h)} = \pi_{j+l-1} + \sum_{h=1}^{l-1}\pi_h\pi_j^{(l<h)}$ $j = 1,2, \ldots$ and $\pi_j^{(1)} = \pi_j$.

Therefore, the h-step ahead forecasts are also the linear combinations of the past observations. Once again, we can write

$$\hat{z}_T(h) = \pi_1{}^{(h)}(y_T + \omega_{SLS}) + \pi_2{}^{(h)} z_{T-1} + \cdots$$

Since $z_{T-1}, z_{T-2}, \ldots$ are already outlier free, so,

$$\hat{z}_T(1) = \pi_1{}^{(h)}\hat{y}_T(1) + \pi_1{}^{(h)}\omega_{SLS} \ \ldots\ldots\ldots\ldots\ldots \qquad (9)$$

All the forecasts made at future time from the time origin T will be biased because of the effect of outlier at time T. The magnitude of bias will depend upon $\pi_1{}^{(h)}$and the magnitude ω_{SLS}.

The h-step ahead forecast made at time origin $ls = n = T + k$ using z_T is

$$\hat{z}_n(h) = \pi_1{}^{(h)} z_n + \pi_2{}^{(h)} z_{n-1} + \cdots$$

$$\hat{z}_{T+k}(h) = \pi_1{}^{(h)} z_{T+k} + \pi_2{}^{(h)} z_{T+k-1} + \cdots + \pi_{k+1}{}^{(h)} z_T + \cdots \ \ldots\ldots\ldots \qquad (10)$$

In order to get the exact impact of SLS, for $ls = n = T + k$, let the SLS affects in a seasonal manner at time point $T = ls/2$, then equation (10) becomes:

$$\hat{z}_{T+k}(h) = \pi_1{}^{(h)}\big(y_{T+k} + \omega_{SLS}\, S(B) I_{T+k}(T)\big) + \pi_2{}^{(h)}\big(y_{T+k-1} + \omega_{SLS}\, S(B) I_{T+k-1}(T)\big) + \cdots + \pi_{k+1}{}^{(h)}(y_T + \omega_{SLS}) + \pi_{k+2}{}^{(h)} y_{T-1} + \cdots,$$

$$where\ \ S(B) = \sum_{j=0}^{ls-T} B^{jS} - \frac{1}{S-1}\sum_{j=0}^{ls-T}\sum_{i=0}^{S-1} B^{i+jS}\ \ for\ \ ls = n = T + k.$$

Hence, $\hat{z}_{T+k}(h) = \hat{y}_{T+k} + \left(\pi_1{}^{(h)}\omega_{SLS} + \pi_2{}^{(h)}\left(\frac{-1}{s-1}\right)\omega_{SLS} + \pi_3{}^{(h)}\left(\frac{-1}{s-1}\right)\omega_{SLS} + \cdots + \pi_{k+1}{}^{(h)}\omega_{SLS}\right)$

$$\hat{z}_{T+k}(h) = \hat{y}_{T+k} + \left(\sum_{j=0}^{k}\pi_{1+js}{}^{(h)}\,\omega_{SLS} + \left(\frac{-1}{s-1}\right)\sum_{j=1}^{k}\sum_{i=0}^{s-1}\pi_{js-i}{}^{(h)}\,\omega_{SLS}\right) \ldots\ldots\ldots \quad (11)$$

The forecast made at some future time origin will also be contaminated by the outlier, *i.e.*, SLS at time T. Moreover, the effect of SLS when forecasting from any origin away from T, *i.e.,* T+k will contaminate every lead time forecast.

If the effect of SLS is not corrected for, then subsequently forecast may be badly biased, therefore, it is important to develop methods to adjust the time series when affected by SLS. If SLS has occurred at time $T; k$ period prior to forecast origin we can write the forecast error as:

$$z_{T+k+h} - \hat{z}_{T+k}(h) = z_{T+k+h} - \hat{z}_{T+k}(h)$$

$$= y_{T+k+h} - \hat{y}_{T+k}(h) + \omega_{SLS}\left[\sum_{j=0}^{k}\pi_{1+js}^{k} - \left(\frac{1}{s-1}\right)\sum_{j=1}^{k}\sum_{i=0}^{s-1}\pi_{js-i}\right]$$

$$z_{T+k+h} - \hat{z}_{T+k}(h) = e_{T+k}{}^{(h)} + \omega_{SLS}\left[\sum_{j=0}^{k}\pi_{1+js}^{h} - \left(\frac{1}{s-1}\right)\sum_{j=1}^{k}\sum_{i=0}^{s-1}\pi_{js-i}\right] \quad (12)$$

where: $e_{T+k}{}^{(h)} = a_{T+k+h} + \psi_1 a_{T+k+h-1} \ldots\ldots + \psi_{h-1} a_{T+k+h-1}$ with $\psi_j\ ; j = 1,2,3,\ldots$ being the coefficients on the basis of eq (2).The mean square of the h-step ahead forecast error in equation (11) is as follows:

$$\text{MSFE (h;k,}\ \omega\ ) = \sigma_a^2 \sum_{j=0}^{h-1}\psi_j^2 + \omega_{SLS}^2\left[\sum_{j=0}^{k}\pi_{1+js}^{(h)} - \left(\frac{1}{s-1}\right)\sum_{j=1}^{k}\sum_{i=0}^{s-1}\pi_{js-i}^{(h)}\right]^2 \ldots \quad (13)$$

And the relative increase in this mean square error that is due to SLS at any time T is given by:

$$\text{IMSFE (h;k,}\ \omega\ ) = \left(\frac{\omega}{\sigma}\right)^2 \frac{1}{\sum_{j=0}^{h-1}\psi_j^2}\left[\sum_{j=0}^{k}\pi_{1+js}^{(h)} - \left(\frac{1}{s-1}\right)\sum_{j=1}^{k}\sum_{i=0}^{s-1}\pi_{js-i}^{(h)}\right]^2 \ldots\ldots\ldots \quad (14)$$

To achieve better understanding, we will apply it to certain stochastic processes i-e SAR(1) and SMA(1).

The SAR(1) or SARIMA $(0,0,0)(1,0,0)_s$ model is given by

$$y_t - \Phi y_{t-s} = a_t \quad Or\ (1 - \Phi B^s)\mathrm{y}_t = a_t$$

We may rewrite this model as $\pi(\mathrm{B})\mathrm{y}_t = a_t$ such that $\pi(\mathrm{B}) = \pi_0 - \pi_1 - \cdots$

where: $\pi_0 = 1, \pi_1 = \cdots = \pi_{s-1} = 0 = \pi_{s+1} = \pi_{s+2} = \cdots$

and $\pi_s = \Phi$ so $\pi_j^{(1)} = \pi_j, \forall j \neq s \quad \pi_j^{(h)} = 0\ \forall j \neq s \quad \pi_s^{(1)} = \Phi$ and $\pi_s^{(h)} = \pi_s^{h-s+1} = \Phi^{h-s+1}\ where\ h\ is\ a\ multiple\ of\ s.$

For $\psi(B) = \frac{1}{1-\Phi B^s} = 1 + \Phi B^s + (\Phi B^s)^2 + (\Phi B^s)^3 + \cdots$

Such that Ψ_0=1, $\Psi_s = \Phi, \Psi_{2s} = \Phi^2, \ldots,\ moreover,\ \Psi_1 = \Psi_2 = \cdots = \Psi_{s-1} = 0 = \Psi_{s+1} = \cdots$

Hence $\Psi_j = \Phi^{j/s}; j = 0, s, 2s, 3s$ ....

Now, $\frac{1}{\sum_{j=0}^{h-1}\psi_j^2} = \frac{1}{1+\Phi^2+\Phi^4+\cdots+\Phi^{2(h-1)}} = \frac{1-\Phi^2}{1-\Phi^{2h}}$

And

$\sum_{j=0}^{k}\pi_{1+js}^{h} = 0, \sum_{j=1}^{k}\sum_{i=0}^{s-1}\pi_{js-i} = \pi_s^{(h)} = \Phi^{h-s+1}.$

Finally, *IMSFE (h; k, ω)* = $\left(\frac{\omega}{\sigma}\right)^2 \left(\frac{1-\Phi^2}{1-\Phi^{2h}}\right) \left[-\left(\frac{1}{s-1}\right)\Phi^{h-s+1}\right]^2$

IMSFE (h;k, ω) = $\left(\frac{\omega}{\sigma}\right)^2 \left(\frac{1}{s-1}\right)^2 \Phi^{2(h-s+1)} \left(\frac{1-\Phi^2}{1-\Phi^{2h}}\right)$ .... (15)

For SMA(1) or SARIMA (0,0,0)(0,0,1)s model $y_t = a_t - \Theta a_{t-s}$ or $y_t = (1-\Theta B^s)a_t$
We may rewrite this model as $\pi(B)y_t = a_t$ such that $\pi(B) = \frac{1}{1-\Theta B^s} = 1 + \Theta B^s + (\Theta B^s)^2 + (\Theta B^s)^3 + \cdots$

where: $\pi_0 = 1, \pi_1 = \cdots = \pi_{s-1} = 0 = \pi_{s+1} = \pi_{s+2} = \cdots$

And $\pi_s = \Theta, \pi_{2s} = \Theta^{2s}, \pi_{3s} = \Theta^{3s}, \ldots$

hence, $\pi_j = \Theta^{\frac{j}{s}}; j = 0, s, 2s, 3s$ *so* $\pi_j^{(1)} = \pi_j, \forall j \neq s$ *,* $\pi_j^{(h)} = 0,\ j = 1,2,3 \ldots$ *and* $\pi_j^{(h)} = \Theta^{\frac{j}{s}}; j = 0, s, 2s, 3s \ldots, \pi_s^{(1)} = \Phi,$

$\pi_s^{(h)} = \pi_s^{h-s+1} = \Theta^{h-s+1}, \pi_{2s}^{(h)} = \Theta^{h-2s+1}, \pi_{3s}^{(h)} = \Theta^{h-3s+1}$ ...

and $\pi_{s+1}^{(h)} = \Theta^{h+s}, \pi_{2s+1}^{(h)} = \Theta^{h\pm 2s}, \pi_{3s+1}^{(h)} = \Theta^{h+3s},$ *where h is a multiple of s*

For $\psi(B) = 1 - \Theta B^s$*, such that* $\Psi_0 = 1$*,* $\Psi_s = 0, \ldots$ *, moreover,* $\Psi_1 = \Psi_2 = \cdots = \Psi_{s-1} = 0 = \Psi_{s+1} = \cdots$.

Now, $\frac{1}{\sum_{j=0}^{h-1}\psi_j^2} = \frac{1}{\Theta^2}$ *and* $\sum_{j=0}^{k}\pi_{1+js}^{h} = \sum_{j=0}^{k}\Theta^{h+js}, \sum_{j=1}^{k}\sum_{i=0}^{s-1}\pi_{js-i} = \sum_{j=1}^{k}\Theta^{js}$.

Finally, the relative increase in mean square error due to SLS in SMA(1) model is when h is multiple of S

IMSFE (h; k, ω) = $\left(\frac{\omega}{\sigma}\right)^2 \left(\frac{1}{\Theta^2}\right) \left[\sum_{j=0}^{k}\Theta^{h+js} - \left(\frac{1}{s-1}\right)\sum_{j=1}^{k}\Theta^{js}\right]^2$

IMSFE (h; k, ω) = $\left(\frac{\omega}{\sigma}\right)^2 \left[\sum_{j=0}^{k}\Theta^{h+js-1} - \left(\frac{1}{s-1}\right)\sum_{j=1}^{k}\Theta^{js-1}\right]^2$ (16)

When h is not a multiple of S

IMSFE (h;k, ω) = $\left(\frac{\omega}{\sigma}\right)^2 \left(\frac{1}{\Theta^2}\right)\left[-\left(\frac{1}{s-1}\right)\sum_{j=1}^{k}\Theta^{js}\right]^2$

IMSFE (h;k, ω) = $\left(\frac{\omega}{\sigma}\right)^2 \left(\frac{1}{s-1}\right)^2 \sum_{j=1}^{k}\Theta^{2(js-1)}$ .... (17)

Using the results in equation (14), (15) and (16), we can conclude that, in the case of SAR (1), the SLS produces an identical relative increase in the mean square of the h-step ahead forecast error for any value of k, *i.e.*, it is invariant regardless of whether the outlier occurs at the forecast origin or at any other preceding time period. However, in the case of SMA(1) model; the longer the time between the time of SLS and the forecast origin (k), the lower would be the inaccuracy caused in point forecasts. In case the forecast lead time (h) is a multiple of S; there is no impact of h on the forecast inaccuracy while for otherwise values of h, the greater the forecast lead time is; the smaller will be the increase in forecast error. Finally, it is noted that the impact of SLS on forecasts depends upon the model parameters in both models.

3.2. SLS Impact on Point and Interval Forecasts under Estimated SARIMA Model Coefficients

In the last section, we explore the impact of SLS on point forecasts obtained from SARIMA models when their parameters are known. But this situation is generally not true. It is observed by Asghar and Urooj (2017), Urooj (2016; unpublished PhD thesis) that SLS affects the parameter estimates resulting in bias in the estimates produced. Hence, ignoring SLS will have exacerbated effect on point forecast and prediction intervals.
Using the basic assumption about the error term for SAR $(1)_s$ for s=12 model

(i) $E(a_t) = 0$,

(ii) $E({a^2}_t) = \sigma_a^2$,

(iii) $E(a_t a_{t-k}) = 0; t \neq s$,

(iv) $E(y_t a_t) = 0$.

For the model $y_t = \Phi y_{t-s} + a_t$ or $y_t = \Phi y_{t-12} + a_t$ and the OLS estimator for the parameter Φ from the uncontaminated series is $\widehat{\Phi} = \frac{\sum_{t=s+1}^{n} y_t y_{t-s}}{\sum_{s+1}^{n} y_t^2}$ and for the series from eq(4) $z_t = y_t + \omega_{SLS}\, S(B) I_t(T)$ it is given by $\widehat{\Phi} = \frac{\sum_{t=s+1}^{n} z_t z_{t-s}}{\sum_{s+1}^{n} z_t^2}$ or $\widehat{\Phi} = \frac{\sum_{t=s+1}^{n}(y_t + \omega_{SLS}\, S(B) I_t(T))(y_{t-s} + \omega_{SLS}\, S(B) I_{t-s}(T))}{\sum_{s+1}^{n} z_t^2}$

Approximation to $E(\widehat{\Phi})$ for SAR (1) processin the presence of SLS is given by $E(\widehat{\Phi}) = \frac{\Phi + A}{1+A}$ where: A=$\frac{2\omega \sum_{s+1}^{n} y_{t-s} v_t}{\sum_{s+1}^{n} {y_{t-s}}^2}$. With the IMFSE for SAR (1) or SARIMA $(0,0,0)(1,0,0)_s$ model

$$IMSFE\,(h; k, \omega\ ) = \left(\frac{\omega}{\sigma}\right)^2 \left(\frac{1}{s-1}\right)^2 \left(\frac{(\Phi+\mathrm{A})^{2(h-s+1)}(1-\Phi+2\mathrm{A})}{(1+A)^{2(2-s)}((1+A)^{2h}-(\Phi+\mathrm{A})^{2h})}\right) \ldots\ldots\ldots\ldots\ldots \quad (18)$$

The second quantity in the numerator of the third term represents the effect on parameter estimates due to SLS. This effect forms a carry-over impact due to the nature of outlier. In order to learn more about the impact of SLS, in our study for SAR (1) and SMA (1) process, we replace the parameter estimates $\widehat{\Phi}$ and $\widehat{\Theta}$ by their biased expected values in equation (15) and (16) respectively. These increases are similar to the ones which are obtained under the assumption that the parameters are known. This is done via simulations and is discussed in detail in the next section.

4. Simulation Analysis

The R package will be used with different libraries, especially TSA, forecast, seasonal, etc., for simulation study in CRAN-R codes. In the first step, we used the data generating process of SARIMA, generate three series, with SLS, without SLS, and adjusted for SLS, in the second step we estimate three models, respectively, and will record each model parameter estimate, and standard error. Further, in the third step, forecast error and one step ahead forecast interval is recoded for each model using 5000 iterations on SARIMA $(0,0,0)(1,0,0)_4$

and (0,0,0)(0,0,1)$_4$ models and results are compared with, without SLS and adjusted for SLS series in terms of model estimation and diagnostics, one step ahead forecast and interval forecasts and a number of forecast error measures.

4.1. Impact of SLS on Forecast Accuracy

The impact of SLS is examined for two cases, namely SAR (1) and SMA (1). We observe that the existence of $\omega_{SLS} = 5\sigma$ causes substantial bias in the estimation of model parameters. We initially estimate the SARIMA (0,0,0)(1,0,0)$_4$ for $\Phi_4 = \{0.2, 0.4, 0.6, 0.8\}$ and SARIMA(0,0,0)(0,0,1)$_4$ for $\Theta_4 = \{0.2, 0.4, 0.6, 0.8\}$ model with SLS at time T and observe the impact of SLS on forecast by calculating one step ahead forecast for 4 quarters. We calculate several statistics using the simulated sampling distribution of estimators. These include the parameter estimates, their sampling distributions and standard errors, root mean square errors, mean absolute errors, autocorrelation function and mean absolute square error are calculated.

4.1.1. The SAR (1) Model

We observed that series with SLS of magnitude $\omega_{SLS} = 5\sigma$ caused biased in sampling distribution of $\widehat{\Phi}_4$. $E(\widehat{\Phi_4})$ remain between [0.189, 0.768] in case of series free of outlier and between [0.880, 0.924] series with outlier. However, when adjusted the series for outlier $E(\widehat{\Phi_4})$ remain between [0.5508, 0.825]. The sampling distribution of $\widehat{\Phi}_4$ series with SLS is 33 % to 17 % high compared to the sampling distribution of $\widehat{\Phi}_4$ the series adjusted for SLS. However, in the case of series free of SLS the sampling distribution of $\widehat{\Phi}_4$are approximately unbiased. For all values SLS cause bias which reduce through adjustment of SLS.

For the series with SLS the sampling distribution of $\widehat{\Phi}_4$ yield high $E(\widehat{SE})$, $E(\widehat{RMSE})$, $E(\widehat{MAE}), E(\widehat{ACF})$ and $E(\widehat{MASE})$. While series adjusted for SLS it become minimum. However, the series without SLS have minimum $E(\widehat{SE})$, $E(\widehat{RMSE})$, $E(\widehat{MAE}), E(\widehat{ACF})$ and $E(\widehat{MASE})$ compared to the series with SLS and adjusted for SLS. The sampling distribution of $\widehat{\Phi}_4$ series with SLS have $E(\widehat{SE})$ 5 % high compared to the series adjusted for SLS. Similarly, $E(\widehat{RMSE})$[9] is 12% to 23%. And $E(\widehat{MAE})$[10] is 13% to 5%, high. The sampling distribution of $\widehat{\Phi}_4$ series with SLS have $E(\widehat{ACF})$[11] is approximately the same and negative, but when series adjusted for SLS, the ACF of Φ_4 values increase and become positive. Similarly, $E(\widehat{MASE})$[12] is minimum in case of with outlier series while in case of no SLS it is relatively greater and series 'adjusted for SLS' have high values (see Table 1).

[9] *Mean value of Root mean square error (RMSE).*
[10] *The average value of Mean Absolute Error (MAE).*
[11] *Mean value of Auto Correlation Function.*
[12] *Mean value of Mean Absolute Scale Error.*

Table 1. Sampling Distribution of SAR-Hat $\widehat{\emptyset_4}$ and Forecast Accuracy

	n=50 ,	W=5sigma,		cv=3.5		BP= 7	
Series with SLS							
SAR1	Coeff	SE	ME	RMSE	MAE	ACF	MASE
∅=0.2	0.880	0.088	0.0243	1.317	1.021	-0.015	0.287
∅=0.4	0.890	0.082	0.0247	1.238	0.956	-0.012	0.265
∅=0.6	0.908	0.077	0.0518	1.149	0.912	0.017	0.244
∅=0.8	0.924	0.073	0.0249	1.120	0.858	-0.002	0.221
Series Adjusted for SLS							
SAR1	Coeff	SE	ME	RMSE	MAE	ACF	MASE
∅=0.2	0.5508	0.081	-0.009	1.150	0.919	0.0342	0.572
∅=0.4	0.6412	0.077	-0.0122	1.099	0.878	0.0323	0.528
∅=0.6	0.7349	0.074	-0.0179	1.045	0.835	0.0135	0.463
∅=0.8	0.825	0.072	-0.0187	1.032	0.821	0.0338	0.384
Series without SLS							
SAR1	Coeff	SE	ME	RMSE	MAE	ACF	MASE
∅=0.2	0.189	0.0685	1.1E-06	0.983	0.778	-0.005	0.689
∅=0.4	0.381	0.0687	5.5E-06	0.983	0.779	-0.006	0.646
∅=0.6	0.574	0.0686	3.6E-05	0.984	0.780	-0.004	0.569
∅=0.8	0.768	0.0685	1.0E-04	0.984	0.781	-0.004	0.437

Note: n is the number of samples have four quarter so total observation 50*4. C.v* is the critical value for detecting of SLS. BP* is the break point where SLS occur. All measures have mean value in the table getting from 5000 number of iterations.*

4.1.2 The SMA (1) Model

We observed that series with SLS of magnitude $\omega_{SLS} = 5\sigma$ caused bias in sampling distribution of $\widehat{\Theta}_4 . E(\widehat{\Theta}_4)$ value is between [0.194, 0.797] in case of series free of outlier and is between [0.6912, 0.929] series with outlier, and falls between [0.375, 0.957] when adjusted for outlier. The sampling distribution of $\widehat{\Theta}_4$ series with SLS is 25 % to 17 % high compared to the sampling distribution of $\widehat{\Theta}_4$ the series adjusted for SLS. However, in the case of series free of SLS the sampling distribution of $\widehat{\Theta}_4$ are approximately unbiased. For all values SLS cause bias which reduce through adjustment of SLS.

For the series with SLS the sampling distribution of $\widehat{\Theta}_4$ yield high $E(\widehat{\boldsymbol{SE}})$, $E(\widehat{RMSE})$,$E(\widehat{MAE}), E(\widehat{ACF})$ and $E(\widehat{MASE})$, but it becomes very small when adjusted for SLS. However, the series without SLS have minimum $E(\widehat{SE})$, $E(\widehat{RMSE})$, $E(\widehat{MAE})$, $E(\widehat{ACF})$ and $E(\widehat{MASE})$ compared to the series with SLS and adjusted for SLS. The sampling distribution of $\widehat{\Theta}_4$ series with SLS have $E(\widehat{SE})$ 5 % high as compared to the series adjusted for SLS. Similarly, $E(\widehat{RMSE})$ is 55 % to 75 % and $E(\widehat{MAE})$ is 58 % to 47 % high. The sampling distribution of $\widehat{\Theta}_4$ 'series with SLS' have $E(\widehat{ACF})$ is higher than its band 0.05 line, but when 'series adjusted for SLS' the ACF of $\widehat{\Theta}_4$ values become smaller and become below band line 0.05 its first lag. Similarly, $E(\widehat{MASE})$ is minimum in case of 'with outlier series' while in case of 'no SLS' series and 'adjusted for SLS' series have high values (see Table A1 in Appendix[13]).

[13] *The tables in Appendix are available online.*

4.2 Impact of SLS on Forecast for Different Sample Sizes

We generate two series of sample size 50 and 100 from data generating process (DGP) following 2 cases; firstly for SAR (1) with Φ_4 values [0.2, 0.4, 0.6, and 0.8] and secondly for SMA (1) with $\widehat{\Theta}_4$ values as [0.2, 0.4, 0.6, 0.8]. Forecasting performance in both models for the two sample sizes are observed for 'with SLS' series and series 'adjusted for SLS' series. Results are compared for observing the SLS impact in association with difference of sample size.

4.2.1 Case of SAR (1)

From the result for SAR (1) model we have observed in Table 3 that SLS of magnitude 5σ affecting series with two different sample sizes do not affect the estimated coefficient in the SAR(1) model . Similar behaviour is observed for series 'adjusted for SLS'. Impact of sample size on forecast accuracy is not significant in SAR (1) model. We conclude that in series with small sample sizes, the large sized SLS causes upward bias in the sampling distribution $\widehat{\Phi}_4$, while SLS adjustment reduces the bias in both sample sizes in the same way.

4.2.2 Case of SMA (1)

For the case of SMA(1), the simulation results in Table 4 show that with an increase in sample size, there is no change in coefficient values, *i.e.*, coefficient estimates remain biased and approximately same at two sample sizes, *i.e.,* n=50 and n=100. Measure of forecast accuracy remains the same and constant. So, we conclude that at all sample sizes large SLS causes similar upward bias in the sampling distribution $\widehat{\Theta_4}$ and similar effect on forecast accuracy measures, except for the SE, which decreases due to increase in sample size (see Table A2 and Table A3 in the Appendix).

4.3 Impact of Various Locations of SLS on Forecast Accuracy

In this section, simulation experiment is used to identify the impact of SLS occurring at different locations in SAR (1) and SMA (1) models, respectively.

4.3.1 Case of SAR (1) Model

From the results we observed that bias of sampling distribution $\widehat{\Phi}_4$ is high when 'SLS occurs at start' of the series, a very small reduction in bias is noticed when 'outlier occurs at the middle' of the series, while bias becomes minimum in the case 'SLS occur at near the forecast origin'. The model diagnostic measures like SE and ME are approximately the same for all three cases. Minor changes in RMSE and MASE while no changes in ACF and MASE are observed due to different outlier location. To conclude, the location of the outlier have no effect on the forecast accuracy measure in the case of SAR (1) model (see Table A4 in the Appendix).

4.3.2 Case of SMA (1)

From the results listed in Table 2 we observe that bias of sampling distribution $\widehat{\Theta}_4$ is high when 'SLS occurs at start' of the series, a very small reduction in bias is noticed when 'outlier occurs at the middle' of the series, while bias becomes minimum in the case 'SLS occur at near the forecast origin'. There are minor changes in model diagnostic measures like SE and ME due to different location of outliers. RMSE major changes is noticed due to different outlier location such as SLS near forecast origin leads to approximately 50 % less RMSE than when 'SLS occur at start of the series' in the SMA (1) model. Similarly, MAE value in the case of 'SLS at the beginning of the series', approximately 61 % to 40 % high in case of 'outlier near forecast origin'. In case of ACF and MASE no changes are noticed due to outlier location. To conclude, the location of the outlier has effect on the forecast accuracy measure in the case of SMA (1) model: SLS occurring at the start of the series have more effect on forecast accuracy measure, while less effect on forecast measure is noticed in case of SLS occurring at the end of the series (see Table 2).

Table 2. SMA-Hat $\widehat{\theta_1}$, Forecast Accuracy, with Different Location of SLS

n=50 , W=5sigma,3.5 BP=7							
SMA	Coeff	SE	ME	RMSE	MAE	ACF	MASE
0.1	0.660	0.128	0.023	2.053	1.618	-0.204	0.456
0.3	0.722	0.122	0.023	1.969	1.550	-0.205	0.276
0.5	0.791	0.116	0.024	1.887	1.484	-0.204	0.256
0.7	0.870	0.110	0.024	1.815	1.424	-0.199	0.234
0.9	0.985	0.104	0.021	1.736	1.357	-0.191	0.204
n=50 , W=5sigma,3.5 BP=25							
SMA	Coeff	SE	ME	RMSE	MAE	ACF	MASE
0.1	0.590	0.105	0.014	1.757	1.336	-0.180	0.529
0.3	0.662	0.101	0.014	1.682	1.277	-0.179	0.276
0.5	0.740	0.096	0.014	1.614	1.221	-0.176	0.256
0.7	0.824	0.093	0.013	1.561	1.179	-0.169	0.234
0.9	0.956	0.088	0.010	1.492	1.121	-0.159	0.204
n=50 , W=5sigma,3.5 BP=38							
SMA	Coeff	SE	ME	RMSE	MAE	ACF	MASE
0.1	0.487	0.088	0.0159	1.477	1.098	-0.142	0.609
0.3	0.576	0.084	0.0155	1.419	1.055	-0.137	0.276
0.5	0.672	0.082	0.0159	1.369	1.017	-0.129	0.256
0.7	0.780	0.080	0.0145	1.332	0.990	-0.118	0.234
0.9	1.018	0.073	0.0119	1.231	0.911	-0.105	0.204

Note: n is the number of samples have four quarter so total observation 50*4, And 100*4. C.v* is the critical value for detecting of SLS. BP* is the break point where SLS occur. All measures have mean value in the table getting from 5000 number of iterations.*

4.4 Impact of SLS on Prediction Interval

For the prediction interval of SAR (1) and SMA (1) model, respectively, series of sample size 50 and 100 are generated under SAR (1) with Φ_4 values [0.2, 0.4, 0.6, and 0.8] and SMA (1) model with $\widehat{\Theta}_4$ values as [0.2, 0.4, 0.6, 0.8] for the two cases, *i.e.*, with SLS and without SLS. Then, one step ahead forecast for both cases is calculated and its 95% prediction interval (predict $\pm$ 2*SD). The processes are iterated 5000 times, then the results are compared for all situations at different sample size.

4.4.1 Case of SAR (1) and SMA (1) Models

We have observed that for the SLS occurring in the first quarter of series, the width of the prediction interval is large followed by the prediction interval of other quarters. While series have no SLS the width of the prediction is more precise and the width is almost the same for every quarter. From the results, we conclude that SLS of magnitude 5σ is very sensitive to prediction interval at all values of SAR (1), SMA (1) parameters.

The prediction interval in case of SLS in SAR(1) model results in forecast interval for the first quarter as [6.9808, 1.8302] followed by the other quarters as [2.5892, -2.5613] ,[2.6101, -2.5403], [2.6101, -2.5679], respectively, while in case of without SLS they are [1.9574, -1.9514], [1.9546, -1.9542] , [1.9555, -1.9532], [1.9555, -1.9558], respectively. Similarly, in case of SMA (1) model with small parameter values, the prediction interval in case of SLS the forecast intervals for the four quarters are [5.9069, -2.4936], [4.3284, -3.4310] ,[4.3335, -3.4258], [4.5791, -3.3671], respectively, while in the case of without SLS they are [1.9543, -1.9486], [1.9518, -1.9511] , [1.9520, -1.9508], [1.9520, -1.9530]. Hence, prediction interval at all parameter values for SAR(1) as well as for SMA (1) model is very sensitive to SLS.

To conclude, series that have SLS is in the first quarter of the series than no SLS, will not only affect the prediction interval of the first quarter, but will also affect the prediction interval of other quarters, or in other words SLS of magnitude 5σ not only affects quarter where it occurs but its effect is also on other quarters of the series, which is confirmed through prediction interval. Comparing the results of prediction interval of SAR (1) and SMA (1) models, results show that the prediction interval of SMA (1) model is more sensitive to SLS as compared to SAR (1) model. This is also confirmed by forecast error measures, which are more sensitive in the SMA (1) model than in the SAR(1) model (see Tables A5 and A6 in the Appendix).

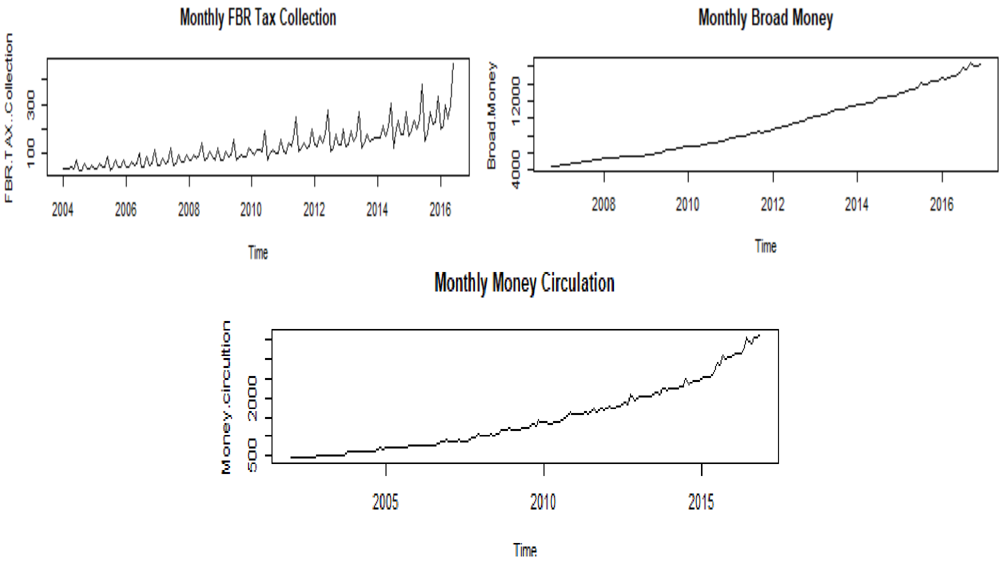
5. Empirical Analysis

This section contains the empirical study conducted for Pakistan using variables measured on monthly frequency, three monthly measured time series, namely Tax collection (2004 M1 to 2016 M6) collected by Federal Board of Revenue, Money in circulation (2002 M1 to 2016 M12) and Broad money (2006 M10 to 2016 M12). The data is taken from Federal Board of Revenue (FBR) annual reports, and International Financial Statistics (IFS). For the identification of outlier type, size and time of occurrence we have run the outlier detection and adjustment procedure suggested by Kaiser and Maravall (2001) modified by Asghar and Urooj (2017) with five possible types of outliers, *i.e*., AO, IO, LS, TC and SLS.

5.1 Graphical Analysis of Raw Series

This section contains graphical analysis of 3 monthly measured time series, *i.e.*, Federal Board of Revenue (FBR) tax collection, broad money, and money circulation. The graphical representation of tax collection indicates seasonality with pair of SLS; one SLS is visualized at $2004M_6$ and second one is readable at $2006M_{11}$. Broad Money has weak seasonality with SLS visible at July 2007, while money circulation plot shows seasonal pattern along with a couple of outliers; SLS at $2009M_{11}$ along with LS at July 2015. Couple of AO outlier is also visible. We notice that we have recognized these outliers just by visualizing the raw data series. It requires mathematical verification, too (see Figure 1).

Figure 1: Graphical Analysis of Raw Series



5.2 Detail of Outliers Detection

In table 3, we find Two SLS along with two AO in the series of FBR monthly Tax collection, one SLS and one LS in money circulation, along with four AO, and two SLS in monthly broad money (for more details see Appendix for Tables A7-A13 and Figures 2-4[14]).

5.3 Impact of Outlier on SARIMA Model

From the results of Tables A9 to Table A11 we observe that the selected model in the case of series 'with SLS' has high standard errors of the estimates, high standard error of the residual and high Akaike information Criteria (AIC). However, after the outlier detection and adjustment for outliers the model obtains low standard error of the estimates, small SE (residuals) value as well as minimum AIC statistic.

From the three models we have applied on three series, we concluded that presence of SLS affects model estimate, standard error, residual standard error and AIC statistic of the model. However, the suggested procedure of Asghar and Urooj (2017) as modification of Kaiser and Maravall (2001) for detection and adjustment of outliers improves the results of the SARIMA models.

[14] *The tables and graphs in appendixare available online as Supplemental material.*

Table 3. Details of Outliers

Type	Index	Time	Size	t-value
Monthly FBR Tax collection				
Type	Index	Time	Size	t-value
AO	36	2006:12	28.65	3.475
SLS	90	2011:06	36.48	3.481
AO	120	2013:12	-81.95	-9.613
SLS	138	2015:06	66.39	6.148
Money in Circulation				
Type	Index	Time	Size	t-value
SLS	95	2009:11	68.65	3.532
AO	116	2011:08	92.08	6.694
AO	130	2012:10	112.81	8.962
AO	151	2014:07	129.08	8.880
LS	163	2015:07	114.17	5.869
AO	165	2015:09	183.98	13.557
Monthly Broad Money				
Type	Index	Time	Size	t-value
SLS	10	2007:07	136.4	5.151
SLS	82	2013:07	187.7	4.431
SLS	120	2016:09	569.8	8.689

5.4. Forecast Accuracy of the SARIMA Models

We assess the performance of the selected SARIMA model accuracy in terms of mean error, root mean square error and mean absolute error, as well as mean absolute scale error and ACF of the models. From results of Table A10, we observe that SARIMA models have high forecast error measures, *i.e.*, mean error, root mean square error, and mean absolute error as well as mean absolute scale error, and ACF of the models, because of outliers in the series. However, when the series is adjusted for the SLS, the forecast error reduces.

From the results, we conclude that SLS leads to poor forecast. However, suggested procedure of Asghar and Urooj (2017) as modification to Kaiser and Maravall (2001) by including IO and SLS in the list of probable outliers along with AO, IO, LS, TC, and SLS for detection and adjustment of outliers improves the results of the SARIMA models and, in turn, improves the forecast errors.

5.5. Impact of SLS on Interval Forecasts of SARIMA Models

In this section, we evaluate the performance of the point and interval forecast of SARIMA models, using different series in the presence of SLS, through interval forecasts and graphical analysis. On the basis of the estimated model we perform one step ahead forecast for 12 months for each series by considering two cases; one ‘with SLS’ and other ‘adjusted for SLS’. After forecast, we get interval forecast for each forecasted month (results in Tables A11-A13).

From the results, we conclude that monthly FBR tax collection, monthly Money Circulation, monthly Broad Money have high standard error for each forecasted month. The standard error of each forecasted month becomes small when outliers are detected and adjusted. High standard error of the series with SLS leads to very large forecast interval; however, after detection and adjustment of SLS results shows more precise prediction intervals (see Appendix, Figure 5). The detailed empirical analysis shows that suggested procedure of Asghar and Urooj (2017) as modification of Kaiser and Maravall (2001) properly detects outliers in monthly time series and improves estimates.

6. Conclusion

In this study, we attempt to examine the consequences of outliers like SLS and other types. The earlier studies by Urooj (2016), Urooj and Asghar (2017), Asghar and Urooj (2017) and Urooj and Asghar (2020) examined the existence, impact and detection of AO, IO, LS, TC and SLS in time series data for various SARIMA(p,d,q)(P,D,Q)$_s$[15] models also by collecting empirical evidence from time series data for Pakistan. However, these studies did not focus on the forecasting performance in the presence of outliers. In this study, we have examined the impact of SLS on point and interval forecasts through simulation experiment and empirically the case of monthly data of Pakistan. We attempt to answer three questions: How does the unrecognized seasonal level shift (SLS) affect the width of the prediction intervals? What is the impact of SLS on forecast for different sample sizes? And how does the impact of outlier vary due to the distance of outlier from the forecast origin?

In order to study the performance of various time series models, we have simulated SARIMA models including SAR (1), *i.e.*, (0, 0, 0) (1, 0, 0)$_4$ and SMA (1), *i.e.*, (0, 0, 0) (0, 0, 1)$_4$ with SLS at T^{th} data point for different parameter values, sample sizes and time of occurrences. We compared the series 'adjusted for SLS' and series 'with SLS' using the modified procedure of Kaiser and Maravall (2001) as suggested by Asghar and Urooj (2017). We extended the strategy suggested by Asghar and Urooj (2017) to the forecasting in the presence of SLS by looking at forecast accuracy and prediction interval. The sampling distribution of $\widehat{\Phi}_4,\widehat{\Theta}_4$ (the parameters of the SAR (1) and SMA(1) parameters, respectively) are studied and a number of statistics, including ME, RMSE, MAE, ACF, MASE and Standard Error (SE), are calculated to measure forecast accuracy. One step ahead forecast and 95 % confidence band as $(\mathrm{Mean} \pm 2\mathrm{SD})$ for SARIMA 'with SLS' and 'without SLS' are calculated and the results compared.

We demonstrate that SLS significantly increases the bias in the SARIMA estimates, increases the inaccuracy of the SARIMA models and significantly affects the prediction intervals. We found that the SLS cause bias in the estimation of model parameter of SAR (1) and SMA (1). However, after detection and adjustment of SLS the bias in parameter estimates is remarkably reduced and forecast accuracy measure and prediction interval significantly improve. Further, the study found that forecast accuracy measures of $\mathrm{SMA}(1)$ and $\mathrm{SAR}(1)$ are very sensitive to SLS and perform poorly due to SLS. However, outlier detection and adjustment procedure reduce the measure of forecast errors, while the model error measures are more sensitive in SMA (1) model as compared to SAR (1) model. Measures of forecast error at different sample sizes are approximately the same in SAR (1) and SMA (1) model such as in sample sizes of $n = 50$ *and* n =100, outlier detection and

[15] *Seasonal Autoregressive Moving Average (SARIMA (p, d, q)(P, D, Q)$_s$) Model, where s, p, d, q, P, D, Q are seasonal frequency order, non-seasonal and seasonal roots, order of integration and seasonal integration of the model as defined by Box and Jenkins (1976). AR(p), MA(q), SAR(P) and SMA(Q) are the special form models defined under SARIMA (p, d, q)(P, D, Q)$_s$.*

adjustment procedure reduce the bias in the coefficient estimate and minimize the forecast error measures in the same way. Further, the study found that the difference of location of SLS from forecast origin had the same effect on bias and forecast accuracy for SAR (1) model, while in SMA (1) model the SLS occurring at the beginning of the series have more effect than outlier occurring at the middle or end of the series due to less bias in the parameter when SLS occurs at the end of the series. In terms of interval forecasts, if SLS is present in the first quarter of the series, then it will not only affect the interval forecasts of the first quarter, but will also affect the interval forecasts of other quarters in SAR (1) as well as SMA (1) models. Furthermore, interval forecasts of SMA (1) model is more sensitive to SLS as compared to SAR (1) model. The empirical study is conducted for Pakistan, using variables on monthly frequency. For the outlier detection and adjustment, we use the procedure suggested by Kaiser and Maravall (2001) and modified by Asghar and Urooj (2017) for five possible types of outliers, *i.e*., AO, IO, LS, TC, and SLS. Study concludes that ignoring SLS causes bias in parameter estimates for SARIMA (p, d, q) (P, D, Q) $_{12}$ and results in high forecast error and interval forecasts. However, when the series is adjusted for SLS, the forecast error measures become smaller and prediction interval improves.

Our study concluded that in economic development, SLS is the outlier that is often not taken into consideration; however, financial and macro indicators exhibit patterns as such. Therefore, it is desirable to examine, identify and capture its impact. SLS have important properties in time series analysis, which are included in Seasonal adjustment software X-13 ARIMA-SEAT developed by the U.S. Census Bureau in collaboration with the Bank of Spain. Outlier detection procedures are suggested in different applications like data cleansing, data-mining tasks and most of the social science research utilizes one of these procedures.

Further study is required for the outlier detection and adjustment, in the form of other time series models, *i.e.*, multivariate models. Furthermore, in the presence of SLS the study of Tsay *et al.* (2000) can be extended to investigate the performance of forecast of the vector autoregressive moving average (VARMA).

7. Limitation of the Study

To carry the study of outlier's detection, adjustment is an important debate found in the literature since many decades. But there is no unanimously theoretical framework which is accepted for the treatment of outliers. Various scholars developed different procedures and very rarely have they developed the same procedure. Our study focuses on the procedure of Maravall (2001), while other alternative procedures, i.e., Hat-matrix, standardize residual approach and DFITS, etc., can also be used for treatment of outliers. The study of Bollen & Jackman (1985) shows that this procedure has been very effective for treatment of outliers. So, we encourage others to not rule out other methods. Further, as argued by Dagum & Bianconcini (2010), the proposed iterative procedure is less vulnerable to spurious and masking effects during outlier detection and allows to jointly estimate the model parameters and multiple outlier effects. The estimation of intervention model allows to verify if there are any insignificant lags to be removed from the model. However, we may also adopt the modification identified by g Li & Chan (2005) by which we can add another step to the procedure as Re-identification of the model where we need to reidentify the ARIMA model underlying the adjusted data series. If the re-identification gives a different model, then repeat the step 2 of outlier detection by using this new model on the original unadjusted data series. Otherwise, terminate the iteration cycle, and the estimated intervention model will be final. This method will be adopted and compared in forthcoming studies.

Disclaimer

Part of this paper is drawn from the first author's M.Phil. thesis stored in the HEC repository of the country.

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