# 1. COPULAS HAVING ZERO-ISOLINE AND ECONOMIC APPLICATIONS[1]

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## Abstract

*In this paper we use some types of copula that do not cover the entire unit square. We use such copula to model the dependence between two random variables that cannot be both lower than given values, but each can be lower than the corresponding value. This can be used in the Phillips curve. Even we consider the values of inflation and unemployment or their rates, there exist $a,b\in\mathbb{R}$ such that the cloud of points around the curve and around the line, respectively, have empty intersection with the set $\left\{(U,V)\in\mathbb{R}^2 \middle| U\le a, V\le b\right\}$, but it is possible to have separately $U\le a$ or $V\le b$.*

**Keywords**: Phillips curve, copula, isolines, zero-isolines

**JEL Classification**: C46, C51, E24, E31.

## 1. Introduction

The Phillips curve (Carnot *et al*., 2005) is a negative relationship, over the long run, between unemployment and the nominal wage. It was exhibited by A. Phillips in 1958 in the case of the United Kingdom. Denoting by $W_t$ the gross nominal wage, by $p_t$ the consumer price index, by $U_t$ the rate of unemployment and by $z_t$ the other explanatory variables, all at the moment *t*, we have the Phillips curve equation

$$\Delta W_t = c + \sum_{i\ge1}\alpha_i\cdot\Delta W_{t-i} + \sum_{i\ge0}\beta_i\cdot\Delta p_{t-i} - \lambda\cdot U_{t-1} + \gamma\cdot\Delta z_t . \qquad (1)$$

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Because in the above formula we have $\lambda > 0$ it results that low unemployment translates into high wage inflation, and vice versa (Carnot *et al*., 2005).

When agents do not suffer from any money illusion the wages are fully indexed by prices, albeit with some lag:

$$1 - \sum_{i \geq 1} \alpha_i = \sum_{i \geq 0} \beta_i \,. \tag{1'}$$

In Albu (2006), the inflation is denoted by $\pi$, and the unemployment by $U$ as above. In order to estimate the natural rate of unemployment, five trend filters are considered: the simple linear trend $(Ye)$, the regress trend $(YTR)$, the loess trend $(YTL)$, the k-smooth trend $(YTK)$, and the Hodrick-Prescott trend $(YHP)$. Using these trend filters the unemployment gaps are:

$$\begin{cases} \Delta Ue = U - Ye \\ \Delta UR = U - YTR \\ \Delta UL = U - YTL \\ \Delta UK = U - YTK \\ \Delta UH = U - YHP \end{cases} . \tag{2}$$

From the graphical representation of the dynamics of inflation and unemployment ($\Delta\pi$ and $\Delta U$) we notice that the points are distributed in the sectors II and IV (in trigonometric sense). The possible differences (i.e., points in the sectors I and III, such that $\Delta\pi \cdot \Delta U > 0$) can be explained by the short-run supply shocks (Albu, 2006).

A copula "couples" the marginal distribution functions to form multivariate distribution functions. Sklar (Sklar, 1959) first used this word in his paper in 1959, almost at the same time as the original article of Phillips on the curve that bears his name (Phillips, 1958). Initially, the study of copulas was involved in the development of the theory of probabilistic metric spaces (Schweizer and Sklar, 1983). Later, more attention is paid to study the dependence structure and the construction of families of multivariate distribution (Nelsen, 1999).

Copulas are now widely applied to a number of fields as econometrics, economics and finance (Bares, Rajna and Gyger, 2001), political science (Flores, 2009), biostatistics (Shih and Thomas, 1995), medical research (Regnate *et al*., 2009), hydrology (Salvadori and De Michele 2007) etc.

Definition 1 (Sungur and Tuncer, 1987; Nelsen, 1991; Flores 2009) *A copula is a function* $C : [0,1]^n \to [0,1]$ *such that*

1) *If there exists* $i$ *such that* $x_i = 0$ *then* $C(x_1, ..., x_n) = 0$.
2) *If* $x_j = 1$ *for all* $j \neq i$ *then* $C(x_1, ..., x_n) = x_i$.
3) C *is increasing in each argument.*

We have the following theorem (Sungur and Tuncer, 1987; Nelsen, 1991; Schweizer, 1991).

**Theorem 1 (Sklar)** *Let* $X_1$, $X_2$,…, $X_n$ *be random variables with the cumulative distribution functions* $F_1$, $F_2$,…, $F_n$, *and the common cdf* $H(x_1,...,x_n)=P(X_1 \le x_1,...,X_n \le x_n)$. *In this case, it exists a copula* $C(u_1,...,u_n)$ *such that* $H(x_1,...,x_n)=C(F_1(x_1),...,F_n(x_n))$. *The copula* C *is well defined on the Cartesian product of the images of the marginals* $F_1$, $F_2$,…, $F_n$.

For any $n-$copula *C* we have (Dall' Aglio, 1991)

$$W(x_1,...,x_n) \le C(x_1,...,x_n) \le \min(x_1,...,x_n), \text{ where:} \quad (3)$$

$$W(x_1,...,x_n)=\sum_{i=1}^{n} x_i - n + 1 \quad (3')$$

is the lower Fréchet bound, and $\min$ is the upper Fréchet bound.

Sometimes we need the overlay probabilities, and we need in this case the notion of co-copula (Văduva, 1994)

$$C^*(u_1,...,u_n)=C(1-u_1,...,1-u_n)+\sum_{i=1}^{n} u_i - n + 1. \quad (4)$$

The probabilistic interpretation of the co-copula is that if $X_1$,..., $X_n$ are random variables with marginals $F_1$,..., $F_n$ and are connected by the copula $C$, we have

$$\overline{H}(x_1,...,x_n)=P(X_1 \ge x_1,...,X_n \ge x_n)=C^*\left(\overline{F_1}(x_1),...,\overline{F_n}(x_n)\right), \quad (4')$$

where: $\overline{F_i}(x_i)=1-F_i(x_i)$.

**Definition 2 (Sungur and Tuncer, 1987; Văduva, 1994; Văduva 2003)** If $n=2$ the copula C is Archimedean if $C(u,u)<u$ for any $u \in (0,1)$, and $C(C(u,v),w)=C(u,C(v,w))$ for any $u,v,w \in [0,1]$. If $n>2$ the copula C is Archimedean if there exists a $n-1$ Archimedean copula $C_1$ and a 2 Archimedean copula $C_2$ such that $C(u_1,...,u_n)=C_2(C_1(u_1,...,u_{n-1}),u_n)$.

Consider a function $\varphi:[0,1]\to\mathbb{R}$ decreasing and convex with $\varphi(1)=0$ and its pseudo-inverse *g* ($g(y)$ has the value *x* if there is *x* such that $\varphi(x)=y$ and *0* in the contrary case). We know (Genest, 1993; Sungur and Tuncer, 1987) that a copula *C* is Archimedean if and only if there is a function $\varphi$ as above, such that for any $x,y \in [0,1]$ we have

$$C(x,y)=g(\varphi(x)+\varphi(y)). \quad (5)$$

In (Vaduva, 1994; Vaduva, 2003) methods to simulate Archimedean copulas are presented.

In the case of the Clayton family, we have for $\theta>0$ (Genest, 1993; Flores, 2009)

$$C(u,v) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{-\frac{1}{\theta}}. \quad (6)$$

For $\theta \to 0$, we obtain the copula *Prod* (independence case), and for $\theta \to \infty$ we obtain the upper Fréchet bound *min*.

From $\frac{\frac{\partial C}{\partial u}}{\frac{\partial C}{\partial v}} = \frac{\varphi'(u)}{\varphi'(v)}$ we obtain first $\varphi'(u) = -u^{-\theta-1}$, and from here

$$\varphi(u) = \frac{u^{-\theta} - 1}{\theta}, \text{ and} \quad (6')$$

$$g(w) = (\theta w + 1)^{-\frac{1}{\theta}}. \quad (6'')$$

Other family of Archimedean copulas presented in (Genest, 1993; Kotz and Seeger 1991; Nelsen, 1991) is the Frank family. In this case, for $\theta \in \mathbb{R}^*$ we have

$$C(u,v) = -\frac{1}{\theta} \cdot \ln\left(\frac{e^{-\theta(u+v)} - e^{-\theta u} - e^{-\theta v} + e^{-\theta}}{e^{-\theta} - 1}\right). \quad (7)$$

We obtain also the copula *Prod* for $\theta \to 0$ and the copula *min* for $\theta \to \infty$. For $\theta \to -\infty$ we obtain the lower Fréchet bound *W*.

From $\frac{\frac{\partial C}{\partial u}}{\frac{\partial C}{\partial v}} = \frac{\varphi'(u)}{\varphi'(v)}$ we obtain first $\varphi'(u) = \frac{\theta e^{-\theta u}}{e^{-\theta u} - 1}$, and from here

$$\varphi(u) = \ln\frac{1 - e^{-\theta}}{1 - e^{-\theta \cdot u}}, \text{ and} \quad (7')$$

$$g(w) = -\frac{1}{\theta}\ln\left(\gamma e^{-w} + 1\right), \text{ where } \gamma = e^{-\theta} - 1. \quad (7'')$$

In the case of the Gumbel-Hougaard family (Flores, 2009; Sungur and Tuncer, 1987; Schweizer, 1991) we have for $\theta \geq 1$ and $\beta = \frac{1}{\theta}$

$$C(u,v) = e^{-\left((-\ln u)^{\theta} + (-\ln v)^{\theta}\right)^{\beta}}. \quad (8)$$

For $\theta = 1$, we obtain the copula *Prod* and for $\theta \to \infty$ we obtain the copula *min*.

From $\frac{\frac{\partial C}{\partial u}}{\frac{\partial C}{\partial v}} = \frac{\varphi'(u)}{\varphi'(v)}$ we obtain first $\varphi'(u) = -\frac{\theta(-\ln u)^{\theta-1}}{u}$, and from here

$$\varphi(u) = (-\ln u)^{\theta}, \text{ and} \quad (8')$$

$$g(w) = e^{-w^{\beta}}. \quad (8'')$$

The Gumbel-Barnett copula is for $0 < \theta \le 1$ (Nelsen, 1991; Flores, 2009)

$$C(u,v) = u \cdot v \cdot e^{-(\theta(\ln u)(\ln v))}. \tag{9}$$

We notice that we have also the copula product (independence) for $\theta \to 0$.

From $\frac{\frac{\partial C}{\partial u}}{\frac{\partial C}{\partial v}} = \frac{\varphi'(u)}{\varphi'(v)}$ we obtain first $\varphi'(u) = -\frac{1}{u(1-\theta \ln u)}$, and from here

$$\varphi(u) = \frac{\ln(1-\theta \ln u)}{\theta}, \text{ and} \tag{9'}$$

$$g(w) = e^{\frac{1-e^{\theta w}}{\theta}}. \tag{9''}$$

The Ali-Mikhail-Haq copula is for $-1 \le \theta \le 1$ (Flores, 2009)

$$C(u,v) = \frac{u \cdot v}{1-\theta(1-u)(1-v)}. \tag{10}$$

We notice that we have the copula product (independence) for $\theta = 0$.

From $\frac{\frac{\partial C}{\partial u}}{\frac{\partial C}{\partial v}} = \frac{\varphi'(u)}{\varphi'(v)}$ we obtain first $\varphi'(u) = -\frac{1}{u(1-\theta(1-u))}$, and from here

$$\varphi(u) = \frac{1}{1-\theta} \cdot \ln\left(\theta + \frac{1-\theta}{u}\right), \text{ and} \tag{10'}$$

$$g(w) = \frac{1-\theta}{e^{(1-\theta)w} - \theta}. \tag{10''}$$

Of course, the above formulae are for $\theta < 1$. The particular case $\theta = 1$ (in this case we have $\frac{0}{0}$) must be treated separately. We have

$$C(u,v) = \frac{uv}{u+v-uv}. \tag{11}$$

From $\frac{\frac{\partial C}{\partial u}}{\frac{\partial C}{\partial v}} = \frac{\varphi'(u)}{\varphi'(v)}$ we obtain first $\varphi'(u) = -\frac{1}{u^2}$, and from here

$$\varphi(u) = \frac{1}{u} - 1, \text{ and} \tag{11'}$$

$$g(w) = \frac{1}{w+1}. \tag{11''}$$

The Nelsen Ten copula is for $0 < \theta \le 1$ (Flores, 2009)

$$C(u,v)=\frac{u\cdot v}{\left(1+\left(1-u^{\theta}\right)\left(1-v^{\theta}\right)\right)^{\frac{1}{\theta}}}. \tag{12}$$

From $\frac{\frac{\partial C}{\partial u}}{\frac{\partial C}{\partial v}}=\frac{\varphi'(u)}{\varphi'(v)}$ we obtain first $\varphi'(u)=-\frac{1}{u\left(2-u^{\theta}\right)}$, and from here

$$\varphi(u)=\frac{1}{2\theta}\cdot\ln\left(2u^{-\theta}-1\right), \text{ and} \tag{12'}$$

$$g(w)=\left(\frac{2}{e^{2\theta w}+1}\right)^{\frac{1}{\theta}}. \tag{12''}$$

The Kendall $\tau$ is (Nelsen, 1991):

$$\tau=P\left(\left(X_1-X_2\right)\left(Y_1-Y_2\right)>0\right)-P\left(\left(X_1-X_2\right)\left(Y_1-Y_2\right)<0\right)=$$
$$4\int_0^1\int_0^1 C(u,v)\frac{\partial^2 C}{\partial u\partial v}dudv-1=1-4\int_0^1\int_0^1\frac{\partial C}{\partial u}\cdot\frac{\partial C}{\partial v}dudv. \tag{13}$$

When the copula is Archimedean and we know the function $\varphi$ in (5) we use the variables change $x=\varphi(u)$ and $y=\varphi(v)$, and finally we obtain (Mereuță, Albu and Ciuiu, 2011)

$$\tau=1-4\cdot\int_0^{\varphi(0)}\int_0^{\varphi(0)-x}\left(g'(x+y)\right)^2 dxdy. \tag{13'}$$

## 2. The construction of copulas with zero-isolines

The graphical representation of the Phillips curve is analogue to the isolines of copula $C_{11}(u,v)=\alpha$. This is the first reason to model the Phillips curve using copulas.

If we consider the copulas in the previous section, we notice that $\lim_{u\to 0}\varphi(u)=\infty$.

From here it results that for any $u,v\in(0,1)$ we have $C(u,v)>0$. Consider now two random variables, *X* and *Y,* that have the marginal cdfs *F*, and *G*, respectively, and their interdependence is modeled by the copula *C* having this property. Denoting by *H* the common cdf of *X* and *Y* we have the property that from $H(x,y)=0$ it results that $F(x)\cdot G(y)=0$. But for the Phillips curve, the involved variables have not this property: each variable can be lower than a given value, but they cannot be lower than the corresponding values in the same time. That is the reason to use other Archimedean copulas such that $\varphi(0)=\lim_{u\to 0}\varphi(u)<\infty$.

Suppose now that $\varphi$ from (5) has the property that $\varphi(0)<\infty$. From

$$\varphi(u)+\varphi(v)>\varphi(0) \tag{14}$$

it results that $C(u,v)=0$, even if $u>0$ and $v>0$. Using the Sklar theorem and replacing *u* by $F(x)$ and *v* by $G(y)$ (*F* and *G* are the marginal cdfs) we obtain the required model.

The zero-isoline of the copula is

$$\{(u, v_u) | C(u,v)=0 \text{ for } v \le v_u;\ C(u,v)>0 \text{ for } v>v_u\}. \tag{15}$$

Using the above definition and (14) we obtain the zero-isoline

$$\varphi(u)+\varphi(v)=\varphi(0). \tag{15'}$$

We can consider now that $\varphi(0)=1$, because we can take in the contrary case

$$\begin{cases} \tilde{\varphi}(u)=\dfrac{\varphi(u)}{\varphi(0)} \\ \tilde{g}(w)=g(w\cdot\varphi(0)) \end{cases}. \tag{16}$$

It results that $\varphi$ is decreasing, convex and we have also $\varphi(0)=1$ and $\varphi(1)=0$. By differentiating the inverse we find that *g* has the same properties. It results in a duality between $\varphi$ and *g*: we can switch them and we obtain a new copula with zero-isoline.

**Example 1** Consider $\varphi(u)=(1-u)^{\theta}$, with $\theta>1$. It results $g(w)=1-w^{\beta}$, where $\beta=\frac{1}{\theta}$.

*The copula is*

$$C(u,v)=1-\left((1-u)^{\theta}+(1-v)^{\theta}\right)^{\beta}.$$

*If we take* $\varphi_1=g$ *we obtain* $g_1=\varphi$*, and*

$$C_1(u,v)=\left(u^{\beta}+v^{\beta}-1\right)^{\theta}.$$

*We notice that the copula $C_1$ is the extension of the Clayton family of copulas on the interval $(-1,0)$. Of course, the value $\theta$ at the denominator in the case of the Clayton copula may confuse us. But, as we have just proved, we can multiply the value of $\varphi$ by any given constant. The other difference is the numerator, but it is the same thing in absolute value. We cannot extend the Clayton copula on $(-\infty,-1]$, because in this case we must have for $C_1$ $\beta\ge 1$. This is impossible, because $\varphi$ is concave in this case.*

In Mereuță, Albu and Ciuiu (2011) the following definition is presented.

**Definition 3** Let $(X,Y)$ *be a bivariate random variable such that the random variables X and Y are connected by the copula C.*

*The copula of non-overlay, non-overlay for* $(X,Y)$ is $C_{11}(u,v)=C(u,v)$.

*The copula of overlay, overlay for* $(X,Y)$ is $C_{00}(u,v)=C^*(1-u,1-v)=C(u,v)+1-u-v$.

*The copula of non-overlay, overlay for* $(X,Y)$ is $C_{10}(u,v)=u-C(u,v)$.

*The copula of overlay, non-overlay for* $(X,Y)$ is $C_{01}(u,v)=v-C(u,v)$.

In the mentioned paper, the classification of competitiveness types was done using the isolines of the copulas from the above definition.

The isolines $C_{11}(u,v)=\alpha$ are, taking into account the relation (5), of the form

$$\varphi(u)+\varphi(v)=\varphi(\alpha). \qquad (17)$$

**Remark 1** The curve that represents the above equation is symmetrical to the line $U=V$, and contains the points $(\alpha,1)$, $(1,\alpha)$ and $(u_\alpha,u_\alpha)$, where $u_\alpha=g\left(\frac{\varphi(\alpha)}{2}\right)$.

Because there is a curve (17) for $\alpha=0$, we say that the considered copula has zero-isoline.

For the isolines $C_{00}(u,v)=\alpha$ we notice first that if for a given pair $(u,v)$ we have $\varphi(u)+(v)<1$ we have the same for any other pair $(u',v')$ with $u'>u$ and $v'>v$. Therefore, the isolines for $C_{00}$ are

$$\begin{cases}\varphi(u)+\varphi(v)=\varphi(u+v+\alpha-1) \text{ if } \varphi(u)+\varphi(v)<1\\ u+v=1-\alpha \text{ if } \varphi(u)+\varphi(v)\geq 1\end{cases}. \qquad (18)$$

When we maximize the value of $\alpha$ in the above equation on the domain $\varphi(u)+\varphi(v)<1$ we obtain the maximum on the zero-isoline. It results that $u+v$ is minimum on this border, hence $u=v=g(0.5)$, and the maximum value of $\alpha$ to consider the first case in (18) is

$$\alpha_{max}=1-2\cdot g(0.5). \qquad (18')$$

If we compute the intersection between $\varphi(u)+\varphi(v)=1$ and $u+v=1-\alpha$ we obtain two points: $(u_1,v_1)$ and $(v_1,u_1)$ with $u_1<v_1$. In the domain $u\leq u_1$ or $u\geq v_1$ with $\varphi(u)+\varphi(v)<1$ we have (from rectangle inclusion) $C_{00}(u,v)<\alpha$. Therefore, we have to find a value of *v* for $u_1<u<v_1$. It is the value of *v* because

$$\begin{cases}\varphi(u)+\varphi(v)-\varphi(u+v+\alpha-1)=\varphi(u)+\varphi(v)-1<0 \text{ if } u+v=1-\alpha\\ \varphi(u)+\varphi(1)-\varphi(u+\alpha)=\varphi(u)-\varphi(u+\alpha)>0\end{cases}. \qquad (19)$$

Denoting by $\alpha_1=\alpha_{max}$ from (18') and by $\alpha_2=C(0.5,0.5)$ we notice from the inclusion of rectangles on the main diagonal that $\alpha_1>\alpha_2$. Therefore, the isolines

$C_{00}(u,v) = \alpha$ have common points with the secondary diagonal $u + v = 1$ for $\alpha \le \alpha_2$. If this relation is without "=", we obtain two points ($(u_2, v_2)$ and $(v_2, u_2)$) such that $u_1 < u_2 < v_2 < v_1$. For "=" we have only one point, $(0.5, 0.5)$, and for the other case ($\alpha_2 < \alpha < \alpha_1$) the isoline is all between the zero-isoline and the secondary diagonal.

For the isolines $C_{10}(u,v) = u - C(u,v) = \alpha$ we obtain

$$\begin{cases} v(u) = g(\varphi(u-\alpha) - \varphi(u)) \text{ for } \alpha < u \le 1 \\ u = \alpha \text{ for } 0 < v \le g(1-\varphi(\alpha)) \end{cases}, \tag{20}$$

hence *v* is increasing on *u*, and we have $v(\alpha) = g(1-\varphi(u))$, and $v(1) = 1-\alpha$.

The isolines $C_{01}(u,v) = v - C(u,v) = \alpha$ are obtained by switching *u* and *v* in the isolines of $C_{10}$.

Next, we present first the way to estimate the parameters of the copula using Kendall's $\tau$, and then we define the isolines $C_{ij}(u,v) = \alpha$ for $i, j \in \{0,1\}$ for the copulas in Example 1.

Consider first $\varphi(u) = (1-u)^{\theta}$. We obtain the lower Fréchet bound *W* for $\theta \to 1$, and the upper Fréchet bound *min,* respectively, for $\theta \to \infty$. Denoting by $\beta = \frac{1}{\theta}$ as above, we have

$$\begin{cases} g'(w) = -\beta \cdot w^{\beta-1} \\ \tau = 1 - 2 \cdot \beta = 1 - \frac{2}{\theta} \end{cases}. \tag{21}$$

To have $\tau < 0$, as the negative relationship between unemployment and inflation suggests, we need $1 < \theta < 2$. We can accept also $\theta = 2$, because the negative relationship has already manifested by the fact that the values of *C* are zero near origin. From the above relation it results that

$$\theta = \frac{2}{1-\tau}. \tag{22}$$

Consider now $\varphi(u) = 1 - u^{\beta}$, where $0 < \beta < 1$. For the limit case $\beta = 1$ we obtain the lower Fréchet bound, *W*. For the limit case $\beta = 0$ we obtain the copula product (the independence case). Denoting by $\theta = \frac{1}{\beta}$ we obtain

$$\begin{cases} g'(w) = -\theta(1-w)^{\theta-1} \\ \tau = -\frac{1}{2\cdot\theta-1} = -\frac{\beta}{2-\beta} \end{cases}. \tag{21'}$$

From the above relation it results that

$$\beta = -\frac{2 \cdot \tau}{1-\tau}. \quad (22')$$

The empirical value of the Kendall $\tau$ is computed using the formula

$$\hat{\tau} = \frac{n_1 - n_2}{C_n^2}, \quad (23)$$

where: *n* is the number of points, $C_n^2 = \frac{n(n-1)}{2}$ is the number of pairs of points having the indexes $i < j$, $n_1$ is the number of these pairs such that $(X_i - X_j)\cdot(Y_i - Y_j) > 0$, and $n_1$ is the number of these pairs such that $(X_i - X_j)\cdot(Y_i - Y_j) < 0$.

The isolines (17) have the equation in the first case ($\varphi(u) = (1-u)^\theta$ for $\theta > 1$)

$$(1-u)^\theta + (1-v)^\theta = (1-\alpha)^\theta. \quad (24)$$

The isolines (18) are

$$\begin{cases} (1-u)^\theta + (1-v)^\theta = (2-u-v-\alpha)^\theta, \text{ if } (1-u)^\theta + (1-v)^\theta < 1 \\ u+v = 1-\alpha, \text{ if } (1-u)^\theta + (1-v)^\theta \geq 1 \end{cases}. \quad (25)$$

The isolines (20) are

$$\begin{cases} v(u) = 1 - \left((1-u+\alpha)^\theta - (1-u)^\theta\right)^{\frac{1}{\theta}} \text{ for } \alpha < u \leq 1 \\ u = \alpha \text{ for } 0 < v \leq 1 - \left(1-(1-\alpha)^\theta\right)^{\frac{1}{\theta}} \end{cases}, \quad (26)$$

and the isolines $C_{01}$ are obtained by switching *u* and *v*.

In the second case ($\varphi(u) = 1 - u^\beta$ for $0 < \beta < 1$) the isolines (17) are

$$u^\beta + v^\beta - \alpha^\beta = 1. \quad (24')$$

The isolines (18) are

$$\begin{cases} u^\beta + v^\beta = 1 + (u+v+\alpha-1)^\beta \text{ if } u^\beta + v^\beta > 1 \\ u+v = 1-\alpha \text{ if } u^\beta + v^\beta \leq 1 \end{cases}. \quad (25')$$

The isolines (20) are

$$\begin{cases} v(u) = \left(1 + (u-\alpha)^\beta - u^\beta\right)^{\frac{1}{\beta}} \text{ for } \alpha < u \leq 1 \\ u = \alpha \text{ for } 0 \leq v \leq \left(1-\alpha^\beta\right)^{\frac{1}{\beta}} \end{cases}, \quad (26')$$

and the isolines $C_{01}$ are obtained by switching *u* and *v*.

## 3. Applications

**Example 2** *Consider the inflation based on prices (CPI=Consumer Price Index) and unemployment rates from January 2006 to December 2010. The data are presented in Appendix A, Table 1.*

*Consider first the marginal distributions being normal, and the data represent the values of the inflation* $(X)$ *and of the unemployment rate* $(Y)$.

The parameters of the above normal distributions (estimated by the moments method) are $E(X)=6.194$ and $Var(X)=2.62132$, and $E(Y)=5.53767$ and $Var(Y)=2.069$, respectively. The estimated value of $\tau$ is $\tau=-0.18249$.

In the case $\varphi(u)=(1-u)^{\theta}$ we obtain, by using (22), $\theta=1.69135$. In the case $\varphi(u)=1-u^{\beta}$ we obtain using (22') $\beta=0.30865$.

If we compute the cdfs $U=F(X)$ and $V=G(Y)$, we obtain the following graphics in $u,v$ coordinates.

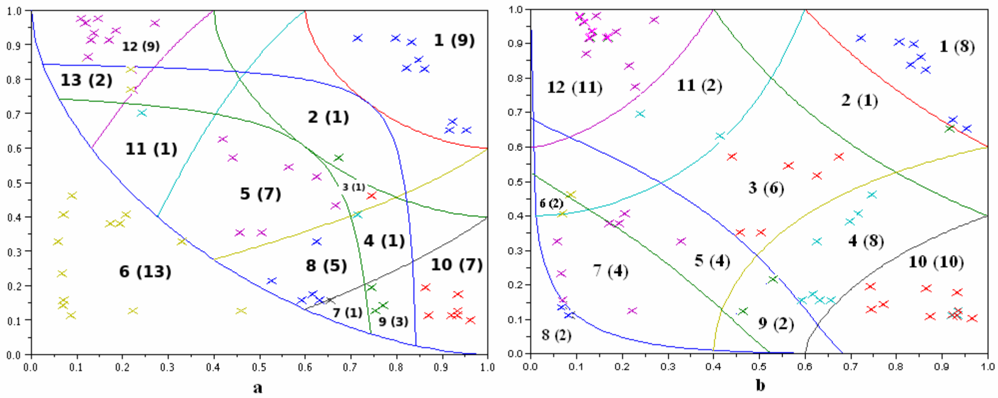



**Figure 1**: The classification of the points $(U,V)$ in the case $\varphi(u)=(1-u)^{1.69135}$ (a), respectively in the case $\varphi(u)=1-u^{0.30865}$ (b), and normal marginals

In the above graphics, the number (index) of the class is represented inside the class, and the corresponding number of points $(U,V)$ in the class is represented between parentheses.

The containing data points are presented in Appendix B, tables 2 and 3.

**Example 3** *Consider in example* 2 *the differences between the values of inflation and unemployment, respectively, of the current month and those of the previous month. We take into account that the values for December 2005 were* $8.6$ *for inflation, and* $5.9$ *for unemployment.*

The parameters of the normal distributions (the difference between two normal distributions is a normal distribution) are $E(X) = -0.01067$ and $Var(X) = 0.37686$, and $E(Y) = 0.01617$ and $Var(Y) = 0.05139$, respectively. The estimated value of $\tau$ is $\tau = -0.04124$.

In the case $\varphi(u) = (1-u)^{\theta}$ we obtain, by using (22), $\theta = 1.92078$. In the case $\varphi(u) = 1 - u^{\beta}$ we obtain, by using (22'), $\beta = 0.07922$.

If we compute the cdfs $U = F(X)$ and $V = G(Y)$, we obtain the following graphics in $u, v$ coordinates.

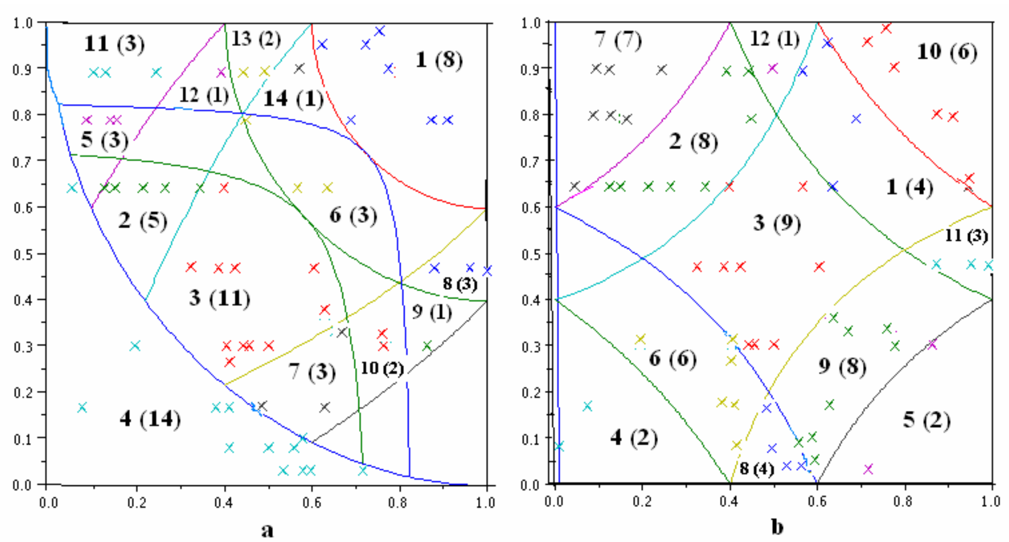



**Figure 2**: The classification of the points $(U, V)$ in the case $\varphi(u) = (1-u)^{1.92078}$ (a) and $\varphi(u) = 1 - u^{0.07922}$ (b) , respectively, and normal marginals for the differences

The containing data points are represented in Appendix B, tables 4 and 5.

**Example 4** *Consider in example* 2 *the log-normal distributions for the marginals.*

Because the methodology used in this paper to estimate the parameter of the copula, $\theta$, does not depend on the marginals, the analytic forms of the copula are the same: $\varphi(u) = (1-u)^{1.69135}$ in the first caseand $\varphi(u) = 1 - u^{0.30865}$ in the second case, respectively. The differences appear in the case of the marginals: because we have to compute the logarithms of the values to obtain normal variables, the expectations of the normal variables become 1.78806 and 1.67909, and their variances become 0.07448 and 0.06551.

If we compute the cdfs $U = F(X)$ and $V = G(Y)$, we obtain the following graphics in $u, v$ coordinates.

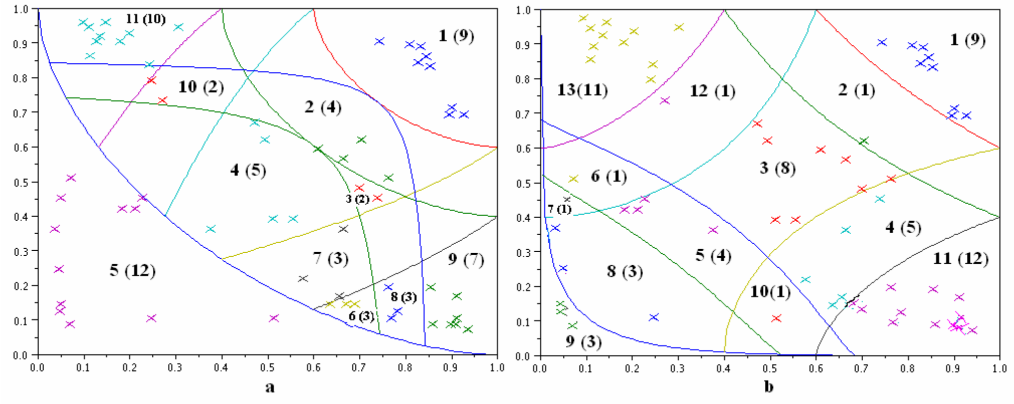
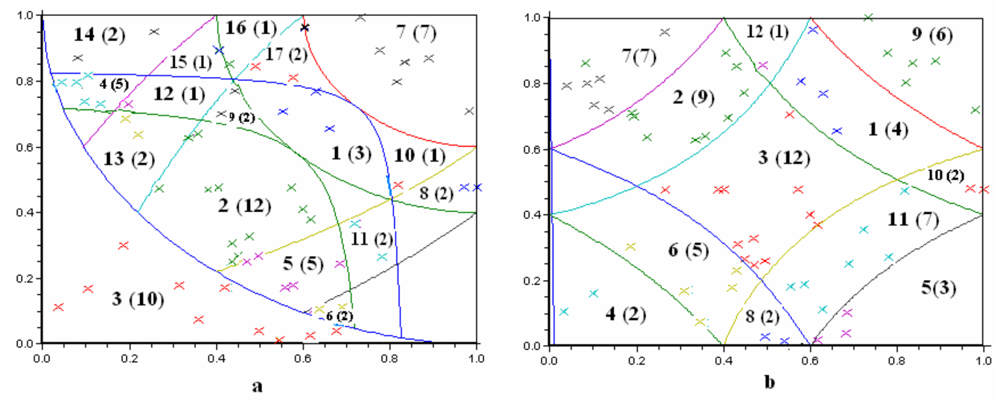



**Figure 3**: The classification of the points $(U,V)$ in the case $\varphi(u)=(1-u)^{1.69135}$ (a) and in the case $\varphi(u)=1-u^{0.30865}$ (b) , respectively, and log-normal marginals

The containing data points are presented in Appendix C, table 6 for the first case, and in table 7, for the second case, respectively.

**Example 5** *Consider, in example* 2, *the log-normal distributions for the marginals, and the ratios of the values of inflation, and unemployment for the current month to those of the previous month.*




**Fiure. 4**: The classification of the points $(U,V)$ in the case $\varphi(u)=(1-u)^{1.91145}$ (a) and $\varphi(u)=1-u^{0.08855}$ (b) , respectively, and log-normal marginals for the ratios

The containing data points are presented Appendix C, tables 8 and 9.

The parameters of the normal distributions (the differences between the logarithms of the inflation and unemployment values) are $E(X)=-0.00129$ and $Var(X)=0.01095$, and $E(Y) = 0.00254$ and $Var(Y) = 0.00178$, respectively. The estimated value of $\tau$ is $\tau = -0.04633$.

In the case $\varphi(u) = (1-u)^{\theta}$ we obtain, by using (22), $\theta = 1.91145$. In the case $\varphi(u) = 1-u^{\beta}$ we obtain, by using (22'), $\beta = 0.08855$.

If we compute the cdfs $U = F(X)$ and $V = G(Y)$, we obtain the following graphics in $u, v$ coordinates.

## 4. Conclusions

In Carnot *et al*. (2005) formula (1) present the possibility to replace the unemployment *U* by its logarithm. This transformation is widely used in finance, when Gaussian linear models such as ARIMA ones cannot be used. When trying to reduce the value of the variable, such as the duration of a telephone call; the probability distribution of the mentioned variable becomes log-normal. Taking into account that the logarithm of the log-normal random variable is a normal one, we find a new reason to study the Phillips curve using the probability theory (if unemployment is not log-normal, the logarithm does not produce normal distribution).

For the classification of competitiveness types using a copula (Mereuță, Albu and Ciuiu, 2011) the families of copulas used are classical (Clayton, Frank, Gumbel-Hougaard and Fréchet). But, as we have mentioned, the copula families that we have to use for the classification of inflation and unemployment must be of the type of those presented in this paper. Of course, Definition 3 is also valid in this case.

For the first copula in Example 1 we have $\tau = 0$ for $\theta = 2$. Taking into account the limit cases of *C* (the Fréchet bounds), it results that there is a value $\theta > 1$ such that Spearman $\rho$ is zero. But, due to the property that $C(u, v) = 0$ in a neighborhood of the origin, there is no $\theta$ such that $C = \mathrm{Prod}$ (independence). It is another counterexample to see that the independence implies $\rho = \tau = 0$, and the reverse implication is false.

In the first case of example 2 we notice first that the main part of the data is in a bandwidth, analogous to Albu (2006). The bandwidth is bordered by the isolines $C(u, v) = 0$ and $C(u, v) = 0.4$: 37 from 60 data points.

The 13 data points from the class six are exceptions $(\varphi(u) + \varphi(v) > 1)$, like the points in the sectors I and III in the case of Albu (2006). The exceptions in our case can be explained by the fact that data are only from Romania, and in the period September 2006-September 2007 Romania joined the European Union. Due to the new possibilities owing to labor migration, we had in this period an artificial decrease of unemployment in Romania.

The "bad" exception to the Phillips curve consists in the first class in the corresponding classification (with high inflation and unemployment). The first three points in this class are the first three months of 2006, the year when the Romanian economy started to move on. These points are in the small group of three points in Figure 1 (a). The other six points in the class are the last six ones in 2010. They can be explained by the unhappy mixture between a government measure that increased inflation (increasing the VAT to $24\%$) and another government measure that increased unemployment (public sector staff restructuring). We notice also that the previous 11 months before the above six months are in the classes 13 (August and September 2009) and 12 (October 2009-June 2010), and they are characterized by low inflation and high unemployment: the public sector staff restructuring started before June 2010, when the Romanian government took the measure of increasing the VAT to $24\%$.

If we consider the second type of copula in example 2 the bandwidth is bordered by the isolines $C(u,v)=0$ and $C(u,v)=0.4$ has more data points $(49)$ and only two exception points such that $\varphi(u)+\varphi(v)>1$ (the eighth class). The bad exception with high inflation and unemployment has the same data points as in the first case, except March for 2006.

If we consider the differences between the normal inflations and unemployment rates (example 3) we notice first that the data points are concentrated between the isolines $C_{01}(u,v)=0.6$ and $C_{10}(u,v)=0.6$: 40 from 60 data points in the first case, and 51 from 60 data points in the second case. Moreover, if we consider in the first case the data points except the 14 from the fourth class (exceptions such that $\varphi(u)+\varphi(v)>1$), we have concentration in the bandwidth bordered by the isolines $C_{01}(u,v)=0.4$ and $C_{10}(u,v)=0.4$: 23 from 46 data points in the first case $(50\%)$, and 27 from 60 data points in the second case $(45\%)$. This concentration along the main diagonal of the unit square in the case of the differences is analogue to the concentration in the sectors II and IV in Albu (2006).

The class with high increase in inflation and unemployment (firs class in the first case and the tenth in the second case) contains in both cases the months of August and October 2007, January 2008, January and November 2009, and January 2010. The first three months are included in this class due to the beginning of the economic crisis in USA, and the increasing unemployment of the Romanians that worked abroad (Spain, Italy). The months of January 2009 and 2010 are in this class due to the same reasons as of the bad exception for the last six months in 2010. November 2009 is in this class due to the government crisis in 2009.

When we use the first type of copula we notice that we have more exceptions under the zero-isolines. This phenomenon is analogue to the estimation of the parameters *a* and *b* in the case of the uniform random variable using the moments method: in this case we can obtain $a > \min_i X_i$ or $b < \max_i X_i$. If we choose not to have such exceptions, we need other methods to estimate the parameter of the copula (for instance, the maximum likelihood method).

In Example 4 we have almost the same classification as in Example 2, except for some other classes' numbers. In the first case, we have the same first class (with high inflation and unemployment), and the classes having indexes 8 and 9 in Example 4 are the same in Example 2, but with indexes 9 and 10. In the second case, we have the same class 5 in both examples, and the class 12 in Example 2 is identical to the class 13 in Example 4 (the class with low inflation and high unemployment).

The explanation of these similarities is that the values of the obtained uniform variables for inflation and unemployment are almost the same. For instance, for the first three months in 2006 we obtain the following values of $(U,V)$: $(0.95345, 0.6533)$, $(0.92365, 0.6788)$ and $(0.91625, 0.6533)$ in the case of normal marginals, and $(0.92706, 0.69314)$, $(0.90069, 0.71509)$ and $(0.8945, 0.69314)$ in the case of log-normal marginals.

In Example 3 there is a difference between normal variables, and, because in the case of log-normal variables the difference is between logarithms, in Example 5 we have the ratios of the values of the current month to those of the previous month. The estimated $\tau$ is the same for the examples 2 and 4: -0.18249. In the case of examples 3 and 5 the values of $\tau$ are a slightly different: -0.04124 in the first case, and -0.04633 in the second case. The explanation is that the cdf is an increasing function, whether the distribution is normal or log-normal in the first case, and the fact that there are four real numbers *a*, *b*, *c* and *d* such that $a-b<c-d$ and $\frac{a}{b}>\frac{c}{d}$.

But we have also analogous similarities between the examples 3 and 5, as for examples 2 and 4. This because, if the values of inflation and unemployment are close, so are their differences. That is why we have not a higher difference between the values of $\tau$.

An open problem is whether there are some economic variables as inflation and unemployment with number higher than *2* such that our model in the present paper can be used.

In (1), some explanatory variables are involved, generically denoted by *z*. A first approach can use a conditional marginal probability distributions for unemployment and inflation. Another open problem is modelling the dependence between these variables and some other variables by other types of copula (eventually having not the properties of the copulas presented in this paper).

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# Appendix A

## The Available Data

**Table 1**

**The Consumer Price Index and the unemployment rate (UR) between January 2006 and December 2010**

| Month | January 2006 | February 2006 | March 2006 | April 2006 | May 2006 | June 2006 |
|---|---|---|---|---|---|---|
| CPI | 8.89 | 8.49 | 8.41 | 6.92 | 7.26 | 7.11 |
| UR | 6.1 | 6.2 | 6.1 | 5.8 | 5.4 | 5.2 |

| Month | July 2006 | August 2006 | September 2006 | October 2006 | November 2006 | December 2006 |
|---|---|---|---|---|---|---|
| CPI | 6.21 | 6.02 | 5.48 | 4.8 | 4.67 | 4.87 |
| UR | 5 | 5 | 4.9 | 5.1 | 5.1 | 5.2 |

| Month | January 2007 | February 2007 | March 2007 | April 2007 | May 2007 | June 2007 |
|---|---|---|---|---|---|---|
| CPI | 4.01 | 3.81 | 3.66 | 3.77 | 3.81 | 3.8 |
| UR | 5.4 | 5.2 | 4.9 | 4.5 | 4.1 | 4 |

| Month | July 2007 | August 2007 | September 2007 | October 2007 | November 2007 | December 2007 |
|---|---|---|---|---|---|---|
| CPI | 3.99 | 4.96 | 6.03 | 6.84 | 6.67 | 6.57 |
| UR | 3.8 | 3.9 | 3.9 | 4.1 | 4.2 | 4.1 |

| Month | January 2008 | February 2008 | March 2008 | April 2008 | May 2008 | June 2008 |
|---|---|---|---|---|---|---|
| CPI | 7.26 | 7.97 | 8.63 | 8.62 | 8.46 | 8.61 |
| UR | 4.3 | 4.3 | 4.2 | 3.9 | 3.8 | 3.8 |

| Month | July 2008 | August 2008 | September 2008 | October 2008 | November 2008 | December 2008 |
|---|---|---|---|---|---|---|
| CPI | 9.04 | 8.02 | 7.3 | 7.39 | 6.74 | 6.3 |
| UR | 3.7 | 3.8 | 3.9 | 4 | 4.1 | 4.4 |

| Month | January 2009 | February 2009 | March 2009 | April 2009 | May 2009 | June 2009 |
|---|---|---|---|---|---|---|
| CPI | 6.71 | 6.89 | 6.71 | 6.45 | 5.95 | 5.86 |
| UR | 4.9 | 5.3 | 5.6 | 5.7 | 5.8 | 6 |

| Month | July 2009 | August 2009 | September 2009 | October 2009 | November 2009 | December 2009 |
|---|---|---|---|---|---|---|
| CPI | 5.06 | 4.96 | 4.94 | 4.3 | 4.65 | 4.74 |
| UR | 6.3 | 6.6 | 6.9 | 7.1 | 7.5 | 7.8 |

| Month | January 2010 | February 2010 | March 2010 | April 2010 | May 2010 | June 2010 |
|---|---|---|---|---|---|---|
| CPI | 5.2 | 4.49 | 4.2 | 4.28 | 4.42 | 4.38 |
| UR | 8.1 | 8.4 | 8.4 | 8.1 | 7.7 | 7.5 |

| Month | July 2010 | August 2010 | September 2010 | October 2010 | November 2010 | December 2010 |
|---|---|---|---|---|---|---|
| CPI | 7.14 | 7.58 | 7.77 | 7.88 | 7.73 | 7.96 |
| UR | 7.5 | 7.41 | 7.35 | 7.08 | 6.95 | 6.87 |

## Appendix B

### The Results in the Case of Normal Marginal Distributions

**Table 2**

**The containing data points in the case of the exact values and the first type of copula**

| Class | Containing points |
|---|---|
| 1 | January-March 2006; July-December 2010 |
| 2 | April 2006 |
| 3 | May 2006 |
| 4 | June 2006 |
| 5 | July and August 2006; February-June 2009 |
| 6 | September 2006-September 2007 |
| 7 | October 2007 |
| 8 | November and December 2007; November 2008-January 2009 |
| 9 | January, September and October 2008 |
| 10 | February-August 2008 |
| 11 | July 2009 |
| 12 | October 2009-June 2010 |
| 13 | August and September 2009 |

**Table 3**

**The containing data points in the case of the exact values and the second type of copula**

| Class | Containing points |
|---|---|
| 1 | January and February 2006; July-December 2010 |
| 2 | March 2006 |
| 3 | April, July and August 2006; March-May 2009 |
| 4 | May and June 2006; October and December 2007; November 2008; January-March 2009 |
| 5 | September-December 2006 |
| 6 | January and February 2007 |
| 7 | March-May and August 2007 |
| 8 | June and July 2007 |
| 9 | September 2007; December 2008 |
| 10 | January-October 2008 |
| 11 | June and July 2009 |
| 12 | August 2009-June 2010 |

**Table 4**

**The containing data points in the case of differences and the first type of copula**

| Class | Containing points |
|---|---|
| 1 | January 2006; August and October 2007; January 2008; January, February and November 2009; January 2010 |
| 2 | February 2006; September and November 2008; April and May 2009 |
| 3 | March, August and November 2006; June, November and December 2007; |
| | May and June 2008; March, September and November 2010 |
| 4 | April-July and September 2006; February-May 2007; April and August 2008; April, May and October 2010 |
| 5 | October 2006; January 2007; October 2009 |
| 6 | December 2006; October 2008 and June 2009 |
| 7 | July 2007; June and December 2010 |
| 8 | September 2007; February 2008; December 2010 |
| 9 | March 2008 |
| 10 | July 2008; August 2010 |
| 11 | December 2008; July 2009; February 2010 |
| 12 | March 2009 |
| 13 | August and September 2009 |
| 14 | December 2009 |

**Table 5**

**The containing data points in the case of the differences and the second type of copula**

| Class | Containing points |
|---|---|
| 1 | January and December 2006; February and December 2009 |
| 2 | February 2006; September and November 2008; March-June and August 2009 |
| 3 | March, August and November 2006; June, November and December 2007; June and October 2008; March 2010 |
| 4 | April and July 2006 |
| 5 | May 2006; March 2008 |
| 6 | June and September 2006; February and March 2007; May 2008; November 2010 |
| 7 | October 2006; January 2007; August and December 2008; July and October 2009; February 2010 |
| 8 | April and May 2006; April 2007; June 2010 |
| 9 | July 2007; July 2008; April, May, August-October and December 2010 |
| 10 | August and October 2007; January 2008; January and November 2009; January 2010 |
| 11 | September 2007; February 2008; July 2010 |
| 12 | September 2009 |

## Appendix C

### The Results in the Case of Log-Normal Marginal Distributions

**Table 6**

**The containing data points in the case** $\varphi(u) = (1-u)^{1.69135}$

| Class | Containing points |
|---|---|
| 1 | January-March 2006; July-December 2010 |
| 2 | April and May 2006; March 2009 |
| 3 | June 2006; February 2009 |
| 4 | July-September 2006; May and June 2009 |
| 5 | October 2006-September 2007 |
| 6 | October and December 2007; November 2008 |
| 7 | November 2007; December 2008; January 2009 |
| 8 | January, September and October 2008 |
| 9 | February-August 2008 |
| 10 | July and August 2009 |
| 11 | September 2009-June 2010 |

**Table 7**

**The containing data points in the case** $\varphi(u) = 1 - u^{0.30865}$

| Class | Containing points |
|---|---|
| 1 | January-March 2006; July-December 2010 |
| 2 | April 2006 |
| 3 | May, July and August 2006; February-June 2009 |
| 4 | June 2006; November and December 2007; December 2008; January 2009 |
| 5 | September-December 2006 |
| 6 | January 2007 |
| 7 | February 2007 |
| 8 | March, April and August 2007 |
| 9 | May-July 2007 |
| 10 | September 2007 |
| 11 | October 2007; January-November 2008 |
| 12 | July 2009 |
| 13 | August 2009-June 2010 |

**Table 8**

**The containing data points in the case of ratios and** $\varphi(u)=(1-u)^{1.91145}$

| Class | Containing points |
|---|---|
| 1 | January and December 2006; October 2008 |
| 2 | February, March, August and November 2006; December 2007; May and June 2008; April 2009; March, September, November and December 2010 |
| 3 | April-July 2006 and September; February-May 2007; April 2008 |
| 4 | October 2006; January 2007; August 2008; October 2009; February 2010 |
| 5 | June 2007; July 2008; April, June and October 2010 |
| 6 | July 2007; May 2010 |
| 7 | August and October 2007; January 2008;<br>January, February and November 2009; January 2010 |
| 8 | September 2007; July 2010 |
| 9 | November 2007; June 2009 |
| 10 | February 2008 |
| 11 | March 2008; August 2010 |
| 12 | September 2008 |
| 13 | November 2008; May 2009 |
| 14 | December 2008; July 2009 |
| 15 | March 2009 |
| 16 | August 2009 |
| 17 | September and December 2009 |

**Table 9**

**The containing data points in the case of ratios and** $\varphi(u)=1-u^{0.08855}$

| Class | Containing points |
|---|---|
| 1 | January and December 2006; February and December 2009 |
| 2 | February 2006; November 2007; September and November 2008; March-June and August 2009 |
| 3 | March, August and November 2006; June and December 2007; June and October 2008;<br>March, June, September, November and December 2010 |
| 4 | April and July 2006 |
| 5 | May 2006; April and July 2007 |
| 6 | June and September 2006; February and March 2007; May 2008 |
| 7 | October 2006; January 2007; August and December 2008;<br>July and October 2009; February 2010 |
| 8 | May 2007; April 2008 |
| 9 | August and October 2007; January 2008; January and November 2009; January 2010 |
| 10 | September 2007; July 2010 |
| 11 | February, March and July 2008; April, May, August and October 2010 |
| 12 | September 2009 |